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On the distribution of extended CIR model

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ABSTRACT

We study the probability distribution of the interest rate in the extended Cox–Ingersoll–Ross model, where all the parameters are time-varying. We show that the distribution can be represented as that of a convergent series of weighted independent central and noncentral chi-square random variables. Simulation algorithms and their applications to finance have been discussed.

1. Introduction

We consider the extended Cox–Ingersoll–Ross term structure model (ECIR model), namely, the spot interest rate $\{r(t)\}_{t\geq 0}$, which satisfies the following stochastic differential equation:

$$\begin{cases} \mathrm{d}r(t) = (-b(t)r(t) + \theta(t))\,\mathrm{d}t + \sigma(t)\sqrt{r(t)}\,\mathrm{d}W(t); \\ r(0) = r_0 \geq 0, \end{cases} \tag{1.1}$$

where $b(t) \geq 0$, $\sigma(t) > 0$ and $\theta(t) \geq 0$ are three time-dependent continuous functions and W is a standard Wiener process. The so-called Cox–Ingersoll–Ross term structure model (CIR model), was first introduced in Cox et al. (1985a, b). In its original specification, the speed of mean reversion $b \geq 0$, the volatility $\sigma > 0$ and the parameter $\theta > 0$ are assumed to be constants. In the ECIR model (1.1), the quantity $d(t) = 4\theta(t)/\sigma(t)^2$, called the dimension of $r(t)$, plays a critical role in the expression of the distribution of $r(t)$.

Several features of the CIR model are particularly attractive. Firstly, it can be justified by general equilibrium considerations (e.g. Cox et al., 1985a). Secondly, the CIR model is a nonnegative-valued ergodic process, and possesses a stationary distribution. Cox et al. (1985b) found that its distribution is a noncentral chi-square. Finally, with the help of CIR model, we can derive a closed form formula for the bond price (Heston, 1993; Chou and Lin, 2007). For practitioners in finance, however, the main shortcoming of the constant parameters version of the model is that it cannot reproduce the original term structure of interest rates (see Hull and White, 1990; Keller-Ressel and Steiner, 2008; Yang, 2005 and all the references therein): yield curves can be only normal, inverse, or humped. The ECIR model, however, has enough parameters to be fitted to the original yield curve.

Maghsoodi (1996) shows that the ECIR model can be represented as the squared norm of a pathwise unique generalized d-dimensional Ornstein–Uhlenbeck processes if and only if its dimension $d \equiv 4\theta(t)/\sigma^2(t)$ is a constant integer (see also

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Jamshidian, 1995; Rogers, 1995). As a consequence, the interest rate $r(t)$ follows a noncentral chi-square distribution of d degrees of freedom. Maghsoodi (1996) shows that, when $d \equiv 4\theta(t)/\sigma^2(t)$ is a positive real number, the ECIR model is in fact almost surely equivalent to a pathwise unique rescaled time-changed squared Bessel process; in the most general case when $d(t) = 4\theta(t)/\sigma^2(t)$ is a time-varying function, the ECIR process can be represented as a lognormal process through a stochastic time-change. Moreover in Maghsoodi (1996), the transition density of $r(t)$ when $d(t)$ is a constant is explicitly given, however it is unclear how can the transition density of $r(t)$ be obtained when $d(t)$ is time-varying, based on the stochastic time-varying lognormal process representation. Shirakawa (2002) points out that in the general case the ECIR interest rate $r(t)$ follows a time-varying dimensional squared Bessel process with deterministic time and state change (see also Pitman and Yor, 1982; Yor, 1992), and the marginal moment generating function of a time-varying dimension squared Bessel is obtained by Carmona (1994). However, as Shirakawa (2002) states, "it is difficult to derive the explicit probability distribution of the time-varying dimensional squared Bessel process with time and state change". Brigo and Mercurio (2006) state that no analytical solution to that problem has yet been found.

The assumption that $d(t)$ is a constant seems too strong to be implemented in a real interest rate modeling problem in finance. As a result there is a strong need to motivate the ECIR model, in which all the parameters are time-varying and behave independently. The aim of our paper is then to derive a feasible representation of the ECIR model's marginal transition probability distribution, where $d(t)$ is time-varying. This problem is equivalent to obtaining a representation theorem (convenient for practical purpose) of time-varying dimensional squared Bessel process (with time and state change)'s marginal probability distribution. In our first main result Theorem 2.1, an explicit form of the marginal distribution of the interest rate $r(t)$, in terms of its characteristic function, is obtained. Then in Theorem 3.1 we show that, for each instant $t \geq 0$, the rate $r(t)$ can be represented as the sum of a noncentral chi-square random variable X_0, and of a convergent series of weighted central chi-square random variables $\{X_1, X_2, \ldots\}$. The variables $X_0, X_1, X_2, \ldots$ are all independent. The random variable X_0 has a number of degrees of freedom equal to $d(0)$. When the dimension d is constant, the random variables $\{X_1, X_2, \ldots\}$ are all zero, and we recover the traditional result. Thus the latter random variables represent deviations of the rate from the constant dimension case. This representation extends Maghsoodi (1996)'s result for the case when $d(t)$ is a constant integer.

It is worth mentioning that, our result is obtained under the assumption that d is differentiable. The main ideas used to derive the above results are:

1. We connect the characteristic function of the squared Bessel process (with state change) to the fact that $r(t)$ is some squared Bessel process with time and state change, in order to derive $r(t)$'s characteristic function (see Theorem 2.1).
2. Under the assumption that d is differentiable, we use the integration by parts formula to show that $r(t)$'s characteristic function can be approximated by the sequence of characteristic functions of weighted sums of noncentral chi-squares (see Theorem 3.1).

Finally in Section 4, we take one example to show how Theorems 2.1 and 3.1 are applied to price a real call option on a zero-coupon bond, where the interest rate is modeled by the ECIR process with time-varying dimension. The *R* codes for the implementation of our algorithms are provided.

2. Characteristic function of ECIR model

In this section we provide characteristic function of the ECIR model $r(t)$, $t \geq 0$, given in (1.1). The main result is stated below.

Theorem 2.1. *For $t \geq 0$, the characteristic function of $r(t)$ is given by*

$$\mathbb{E}[e^{i\omega r(t)}] = \exp\left(i\omega\left(\frac{r_0 e^{-\int_0^t b(u)\,\mathrm{d}u}}{1-2i\omega\Sigma(0,t)} + \int_0^t \frac{\theta(s)e^{-\int_s^t b(u)\,\mathrm{d}u}}{1-2i\omega\Sigma(s,t)}\,\mathrm{d}s\right)\right), \tag{2.1}$$

for all $\omega \in \mathbb{R}$, where the bivariate function $\Sigma(s,t) = \frac{1}{4}\int_s^t e^{-\int_v^t b(u)\,\mathrm{d}u}\sigma^2(v)\,\mathrm{d}v$.

Proof. Let $\{X_t^{(\gamma)}\}_{t\geq 0}$ be a squared Bessel process with initial value $X_0^{(\gamma)} = x$ and time-varying dimension $\gamma(\cdot)$. On one hand, from Proposition 3.4 in Carmona (1994), we derive the characteristic function of $X_t^{(\gamma)}$ as: for all $\omega \in \mathbb{R}$,

$$\mathbb{E}[e^{i\omega X_t^{(\gamma)}}] = \exp\left(i\omega\left(\frac{x}{1-2i\omega t} + \int_0^t \frac{\gamma(u)}{1-2i\omega(t-u)}\,\mathrm{d}u\right)\right). \tag{2.2}$$

On the other hand, it is shown that $\{r(t)\}_{t\geq 0}$ in (2.1) is some time-varying dimensional squared Bessel process with time and state changes (see Lemma 2.4 and Corollary 3.1 in Shirakawa, 2002):

$$\{r(t)\}_{t\geq 0} \sim \left\{\nu(t)X_{\tau(t)}^{(\gamma)}\right\}_{t\geq 0}, \tag{2.3}$$