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Complete asymptotic expansions for the density function of t-distribution

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ABSTRACT

The density function of t-distribution with n degrees of freedom is given by the following formula:

$$f_n(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\,\Gamma(\frac{n}{2})}\left(1+\frac{x^2}{n}\right)^{-(n+1)/2}, \qquad x > 0.$$

Ding (1988) obtained the following asymptotic formula in terms of $1/n$:

$$f_n(x) = \varphi(x)\left\{1 + \frac{x^4 - 2x^2 - 1}{4n} + \frac{3x^8 - 28x^6 + 30x^4 + 12x^2 + 3}{96n^2} + O\left(\frac{1}{n^3}\right)\right\}, \quad n \to \infty,$$

which derives the known result $\lim_{n\to\infty} f_n(x) = \varphi(x)$, where $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is the probability density function of the standard normal distribution. In this paper, we develop Ding's result to produce a complete asymptotic expansion.

1. Introduction

The density function of t-distribution (Student's distribution) with p degrees of freedom (SD_p) is given by the formula

$$f_p(x) = \frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi p}\,\Gamma(\frac{p}{2})}\left(1+\frac{x^2}{p}\right)^{-(p+1)/2}, \qquad x > 0. \tag{1.1}$$

Most often, the values of the parameter p are assumed to be positive integers. However, Pinelis (2015) considered the probability density function $f_p(x)$ for all real $p > 0$.

Ding (1998) obtained the following asymptotic formula in terms of $1/n$ (where $n \in \mathbb{N} := \{1, 2, \ldots\}$):

$$f_n(x) = \varphi(x)\left\{1 + \frac{x^4 - 2x^2 - 1}{4n} + \frac{3x^8 - 28x^6 + 30x^4 + 12x^2 + 3}{96n^2} + O\left(\frac{1}{n^3}\right)\right\}, \qquad n \to \infty, \tag{1.2}$$

which derives the known result

$$\lim_{n\to\infty} f_n(x) = \varphi(x), \tag{1.3}$$

where

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \tag{1.4}$$

is the probability density function of the standard normal distribution (SND).

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In this paper, we establish asymptotic expansions for the density function of t-distribution. More precisely, we provide an explicit formula for determining the coefficients $a_j \equiv a_j(x)\,(j \in \mathbb{N})$ (Theorem 2.1) such that

$$f_p(x) \sim \varphi(x) \exp\left(\sum_{j=1}^{\infty} \frac{a_j}{p^j}\right), \qquad p \to \infty.$$

Based on the obtained result, we give a recurrence relation (Theorem 2.2) and an explicit formula (Theorem 2.3) for determining the coefficients $b \equiv b_j(x)\,(j \in \mathbb{N})$ such that

$$f_p(x) \sim \varphi(x) \left(1 + \sum_{j=1}^{\infty} \frac{b_j}{p^j}\right), \qquad p \to \infty, \tag{1.5}$$

which develops Ding's result (1.2) to produce a complete asymptotic expansion.

2. Results

Theorem 2.1. *The density function $f_p(x)$, defined by (1.1), has the following asymptotic expansion:*

$$f_p(x) \sim \varphi(x) \exp\left(\sum_{j=1}^{\infty} \frac{a_j}{p^j}\right), \qquad p \to \infty, \tag{2.6}$$

with the coefficients $a_j \equiv a_j(x)\,(j \in \mathbb{N})$ given by

$$a_j = (-1)^{j-1} \left(\frac{(1-2^{j+1})B_{j+1}}{j(j+1)} + \frac{x^{2j+2}}{2j+2} - \frac{x^{2j}}{2j}\right), \tag{2.7}$$

where B_n $(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ are the Bernoulli numbers.

Proof. The logarithm of gamma function has the following asymptotic expansion:

$$\ln \Gamma(x+t) \sim \left(x + t - \frac{1}{2}\right) \ln x - x + \frac{1}{2} \ln(2\pi) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_{n+1}(t)}{n(n+1)} \frac{1}{x^n}, \qquad x \to \infty, \tag{2.8}$$

see Luke (1969, p. 32), where $B_n(t)$ denotes the Bernoulli polynomials defined by the following generating function:

$$\frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!}. \tag{2.9}$$

Note that the Bernoulli numbers B_n (*for* $n \in \mathbb{N}_0$) are defined by (2.9) for $t = 0$. It is well known that

$$B_n\left(\tfrac{1}{2}\right) = (2^{1-n} - 1)B_n, \qquad n \in \mathbb{N}_0.$$

Using (2.8) and the Maclaurin expansion of $\ln(1+t)$,

$$\ln(1+t) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} t^j, \qquad -1 < t \le 1,$$

we obtain

$$\ln f_p(x) \sim -\ln\sqrt{2\pi} - \frac{x^2}{2} + \sum_{j=1}^{\infty} \frac{a_j}{p^j}, \qquad p \to \infty, \tag{2.10}$$

where

$$a_j = (-1)^{j-1} \left(\frac{(1-2^{j+1})B_{j+1}}{j(j+1)} - \frac{x^{2j}}{2j} + \frac{x^{2j+2}}{2j+2}\right).$$

Clearly, (2.10) can be written as (2.6). The proof is complete.

We find from (2.7) that the first few coefficients a_j are:

$$a_1 = -\frac{1}{4} - \frac{1}{2}x^2 + \frac{1}{4}x^4, \quad a_2 = \frac{1}{4}x^4 - \frac{1}{6}x^6, \quad a_3 = \frac{1}{24} - \frac{1}{6}x^6 + \frac{1}{8}x^8, \tag{2.11}$$