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A note on the spectral gap for general harmonic measures on spheres

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ABSTRACT

In this paper, we consider general harmonic measures $\mu_x^{n,\beta}$ on the unit sphere S^{n-1} in $\mathbb{R}^n$, where $x \in \mathbb{R}^n$ with $0 \leq |x| < 1, \beta \in \mathbb{R}$ and $n \geq 3$. Following the idea in Barthe et al. (2014) , we obtain the lower bound for the spectral gap of $\mu_x^{n,\beta}$. For the harmonic measure ($\beta = 0$), we improve the lower bound of the spectral gap in Barthe et al. (2014) and we also improve those in Milman (2015) for general $\beta \in \mathbb{R}$.

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1. Introduction

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ $(n \geq 3)$ with the geodesic d and μ the normalized Lebesgue measure on S^{n-1}, i.e. $\mu = \sigma_{n-1}/s_{n-1}$, where σ_{n-1} and $s_{n-1} = \frac{n\pi^{n/2}}{\Gamma(1+n/2)}$ are the uniform surface measure and total volume respectively on S^{n-1}. For $x \in \mathbb{R}^n$ with $0 \leq |x| < 1$, let μ_x^n be the probability on S^{n-1} given by

$$d\mu_x^n(y) = \frac{1-|x|^2}{|y-x|^n} d\mu(y), \quad y \in S^{n-1}, \tag{1.1}$$

which is the harmonic measure related to x. This measure characterizes the hitting distribution of the sphere by standard Brownian motion in $\mathbb{R}^n$ (see Durrett, 1984; Kakutani, 1944). In Schechtman and Schmuckenschläger (1995), G. Schechtman and M. Schmuckenschläger proved that μ_x^n with any $|x| < 1$ have a uniform Gaussian concentration. Barthe et al. (2014) proved that μ_x^n satisfies uniform Poincaré inequality. Later Milman in Milman (2015) considered generalized harmonic measures on sphere, whose form is given by:

$$d\mu_x^{n,\beta}(y) = \frac{1}{Z_{n,\beta}} \frac{1}{|y-x|^{n+\beta}} d\mu(y), \quad y \in S^{n-1} \tag{1.2}$$

for $\beta \in \mathbb{R}$. $Z_{n,\beta}$ is a normalizing constant and $\beta = 0$ is the harmonic measure given in (1.1).

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We say that $\mu_x^{n,\beta}$ satisfies a Poincaré inequality if there exists a finite positive constant C such that for any smooth function $f : S^{n-1} \to \mathbb{R}$, it holds

$$C\mathrm{Var}_{\mu_x^{n,\beta}}(f) \le \int_{S^{n-1}} |\nabla_{S^{n-1}} f|^2 d\mu_x^{n,\beta},$$

where $\nabla_{S^{n-1}}$ is the spherical gradient and $|\cdot|$ is the Euclidean norm. Let $\lambda_1(\mu_x^{n,\beta})$ denote the best constant of the above inequality, which is also the spectral gap of $-\mathcal{L}_x^{n,\beta}$ in $L^2(\mu_x^{n,\beta})$. Here the operator $-\mathcal{L}_x^{n,\beta}$ is given as: for any smooth function on S^{n-1},

$$\mathcal{L}_x^{n,\beta} f(y) = \Delta_{S^{n-1}} f(y) - (n+\beta)\nabla_{S^{n-1}}|y-x| \cdot \nabla_{S^{n-1}} f(y), \quad y \in S^{n-1},$$

where $\Delta_{S^{n-1}}$ is the Laplace–Beltrami operator on S^{n-1} and $\cdot$ is the Euclidean inner product.

When $\beta = 0$, Barthe et al. (2014) offered a two sided estimate on $\lambda_1(\mu_x^n)$. Precisely, by a family of orthonormal basis of the tangent space at any point x on S^{n-1}, they reduced the spectral gap of μ_x^n to that of one dimensional diffusion $\nu_{|x|}^n$ on $[0, \pi]$, where

$$\nu_{|x|}^n(d\theta) = (1-|x|^2)\frac{s_{n-2}}{s_{n-1}} \frac{\sin^{n-2}\theta}{(1+|x|^2-2|x|\cos\theta)^{\frac{n}{2}}} d\theta$$

is the image probability of μ_x^n by the mapping $y \to d(y, e_1)$ with $s_{n-1} = \frac{n\pi^{n/2}}{\Gamma(1+n/2)}$. Therefore, based on the estimate for $\lambda_1(\nu_{|x|}^n)$, they obtained

$$\frac{n-2}{2} \le \lambda_1(\mu_x^n) \le n-1. \tag{1.3}$$

Milman in Milman (2015) studied the curvature-dimension of generalized harmonic measure $\mu_x^{n,\beta}$ and as a consequence he offered a lower bound for $\lambda_1(\mu_x^{n,\beta})$ for $\beta \in (-2, 3n-7)$.

In this paper, following the idea in Barthe et al. (2014), we establish the following main theorem:

Theorem 1.1. *Let $\mu_x^{n,\beta}$ and $\lambda_1(\mu_x^{n,\beta})$ be as above. We have*

$$\lambda_1(\mu_x^{n,\beta}) \ge \begin{cases} n-2, & \text{if } \beta \le 2-n \text{ or } \beta > \dfrac{n-5}{n-1}; \\ \dfrac{(\beta+3)(n-1)}{4}, & \text{if } -1 \le \beta \le \dfrac{n-5}{n-1}; \\ \dfrac{n-\beta}{2} - 1, & \text{if } 2-n < \beta < -1. \end{cases}$$

This lower bounds follows immediately from the comparison (2.2) and Proposition 2.1.

Remark 1.2. When $\beta = 0$, combining the lower bound in Theorem 1.1 and the upper bound in (1.3), we have

$$\text{if } n \ge 5, \quad \frac{3}{4}(n-1) \le \lambda_1(\mu_x^n) \le n-1,$$

$$\text{if } 3 \le n < 5, \quad n-2 \le \lambda_1(\mu_x^n) \le n-1.$$

The lower bound is better than that in (1.3) taken from Barthe et al. (2014).

Remark 1.3. When $\beta = 1$ or $\beta = -n$, as proved in Proposition 2.1, we know $\lambda_1(\nu_{|x|}^{n,\beta}) = n-1$. Thereby it follows from the comparison (2.2) that

$$n-2 \le \lambda_1(\mu_x^{n,\beta}) \le n-1.$$

Remark 1.4. When $\beta = -n-2$, based on Theorem 1.1 and Proposition 3.1, we know

$$n-2 \le \lambda_1(\mu_x^{n,\beta}) \le n.$$

Remark 1.5. When $\beta = -2$ and $n \ge 4$, it follows from Theorem 1.1 and Proposition 3.2 that

$$\frac{n}{2} \le \lambda_1(\mu_x^{n,\beta}) \le n-1.$$

And for $n = 3$, we have

$$1 \le \lambda_1(\mu_x^{3,\beta}) \le 2.$$

Limited by the lack of information on the normalizing constant $Z_{n,\beta}$ for general $\beta \in \mathbb{R}$, we are not able to give a general upper bound for $\lambda_1(\mu_x^{n,\beta})$ via the classical variational formula. Therefore in the third section, we will give upper bound on $\lambda_1(\mu_x^{n,\beta})$ for some special β. The next section is devoted to the lower bound of $\lambda_1(\nu_{|x|}^{n,\beta})$.

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