STAPRO: 8265 Model 3G pp. 1–6 (col. fig: NIL)

ARTICLE IN PRESS

Statistics and Probability Letters xx (xxxx) xxx–xxx

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Dynamic optimality in optimal variance stopping problems

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A R T I C L E   I N F O

Article history:
Received 26 July 2017
Received in revised form 27 May 2018
Accepted 29 May 2018
Available online xxxx

MSC:
60G40
60J25
90C20

Keywords:
Dynamic and static optimality
Markov processes
Optimal variance stopping problems

A B S T R A C T

In an optimal variance stopping (O.V.S.) problem one seeks to determine the stopping time that maximizes the variance of an observed process. As originally shown by Pedersen (2011), the variance criterion leads to optimal stopping boundaries that depend explicitly on the initial point of the process. Then, following the lines of Pedersen and Peskir (2016), we introduce the concept of dynamic optimality for an O.V.S. problem, a type of optimality that disregards the starting point of the process. We examine when an O.V.S. problem admits a dynamically optimal stopping time and we illustrate our findings through several examples.

1. Introduction

Optimal variance stopping (O.V.S.) problems are a new class of optimal stopping problems and were originally introduced in Pedersen (2011). Given a continuous time real-valued Markov process $X := (X_t)_{t\geq 0}$, solving an O.V.S. problem means determining the stopping time which maximizes the right-hand side of

$$\mathbb{V}(x) := \sup_{\tau} \mathrm{Var}_x[X_\tau], \tag{1.1}$$

as well as computing the value function $\mathbb{V}(x)$. In the above expression, the supremum is taken on the class of stopping times τ of X, such that $\mathbb{E}_x[X_\tau^2] < \infty$, and the variance operator $\mathrm{Var}_x[X_\tau] := \mathbb{E}_x[X_\tau^2] - \mathbb{E}_x^2[X_\tau]$ is defined with respect to the probability measure $\mathbb{P}_x$, under which X starts at x. The quadratic dependence of the variance on the expectation makes (1.1) different from a standard optimal stopping problem of the type

$$V(x) := \sup_{\tau} \mathbb{E}_x[G(X_\tau)], \tag{1.2}$$

where the expectation of some suitable gain function $G(\cdot)$ enters linearly (see, e.g., Peskir and Shiryaev, 2006, Shiryaev, 1978). An outstanding consequence of this fact is that while in (1.2) the state space of X can be partitioned into one continuation set, where $V > G$, and one stopping set, where $V = G$, and, therefore, these sets are independent of the initial point x of X, in (1.1) this does not hold true anymore. In other words, the stopping boundaries in an O.V.S. problem depend on x, as the results in Pedersen (2011, Sections 3–5) show.

The dependence of the stopping boundaries on the initial point of the process was also observed in Pedersen and Peskir (2016), where some mean–variance optimal stopping problems are studied. The authors referred to the optimality tied to

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https://doi.org/10.1016/j.spl.2018.05.030
0167-7152/

Please cite this article in press as: Buonaguidi B., Dynamic optimality in optimal variance stopping problems. Statistics and Probability Letters (2018), https://doi.org/10.1016/j.spl.2018.05.030.

the initial value of the process as *static* to distinguish it from what they called *dynamic* optimality. This type of optimality arises from the fact that the change of the initial point of the process (and, in turn, the change of the stopping boundary) yields a different optimal stopping time; the new values of the process lead therefore to solve, dynamically in time, new optimal stopping problems, with the aim to stop when there is no more chance to find a stopping time that could return a better value in the future.

To the best of our knowledge, all the literature on O.V.S. problems deals with the static optimality only (see Definition 2.1). In Pedersen (2011), a general method for solving (1.1) is provided and explicit solutions when X is a geometric Brownian motion, a square-root process and a Jacobi diffusion are obtained; in Gad and Pedersen (2015), problem (1.1) is studied under the assumption that X is an exponentiated Lévy process and it is shown that, in some cases, randomized stopping times (namely stopping times with respect to the filtration generated by X and an independent uniform random variable) might solve (1.1) and achieve a higher variance. In Buonaguidi (2015) another solution method to (1.1) is analyzed and is used in Buonaguidi (2018) to solve the examples from Pedersen (2011); moreover, in Buonaguidi (2018) a possible application to trading strategies of an O.V.S. problem is discussed. In Gad and Matomaki (2016), it is shown that for some diffusion processes an optimal solution can be obtained as mixtures of two hitting times and the strong connection between problem (1.1) and the game theory is disclosed.

The novelty of this paper is the study of dynamically optimal stopping times in the O.V.S. problem (1.1) (see Definitions 2.2 and 2.3). On one hand, we show that when X is a non-trivial (i.e., non-deterministic) stochastic process with no killing or absorbing boundaries, then it is dynamically optimal not to stop at all; on the other hand, when killing or absorption are allowed, denoted by ξ the life time of the process after which it remains trapped in the killing or absorbing state, we show that, conditionally to the event $\{\xi < \infty\}$, ξ is dynamically optimal. These results are consistent with the fact that if we solve (1.1) dynamically in time, in the first case, there is no stopping time after which the observed process ceases its variability, while, in the second case, this stopping time exists and is precisely ξ.

This paper is structured as follows: in Section 2, along the lines of Pedersen and Peskir (2016), we define the concepts of static and dynamic optimality, as well as the concept of dynamic optimality conditionally to $\{\xi < \infty\}$, and we provide our main results. In Section 3, we illustrate our findings by means of several examples, including Lévy processes, Brownian motions absorbed at the origin, Bessel processes and population growth and genetic diffusion models. Finally, in Section 4, we provide a financial interpretation of the results obtained in the previous sections.

2. Main results

We begin this section by defining the static and dynamic optimality discussed in the Introduction: we adapt the definitions used in Pedersen and Peskir (2016) to the O.V.S. problem (1.1). In the sequel, we denote by S the state space of X.

Definition 2.1. A stopping time $\tau^\star$ is statically optimal in (1.1) for $x \in S$ given and fixed, if there is no other stopping time τ such that

$$\mathrm{Var}_x[X_\tau] > \mathrm{Var}_x[X_{\tau^\star}].$$

From this definition we see that the statically optimal stopping time coincides with the stopping time that maximizes the right-hand side of (1.1). Then, the optimal stopping times derived in Buonaguidi (2015, 2018), Gad and Matomaki (2016), Gad and Pedersen (2015) and Pedersen (2011) must be understood in the static sense. We observe that the static optimality concerns a given and fixed initial value x of X; in particular, as shown in the aforementioned articles, changing x implies a change of the stopping boundaries (which depend on x) and hence a variation of the statically optimal stopping time (see Section 3.1 for an explicit example). Then, we can consider the situation where (1.1) is approached dynamically in time, in the sense that each new position of the process leads to solve a new O.V.S. problem. The next definition formalizes this concept.

Definition 2.2. A stopping time $\tau^\star$ is dynamically optimal in (1.1) if there is no other stopping time τ such that

$$\mathbb{P}_x\left(\mathrm{Var}_{X_{\tau^\star}}[X_\tau] > 0\right) > 0, \tag{2.1}$$

for some $x \in S$.

Dynamic optimality is therefore equivalent to determining the stopping time $\tau^\star$ after which there is no chance that some variability occurs in the future. Then, as also observed in Pedersen and Peskir (2016), while the static optimality remembers the past through the initial value x of X, the dynamic optimality disregards it and looks ahead only. The next theorem shows that if X is trivial, then it is dynamically optimal to stop at once; instead, if X is a proper stochastic process, then it is never dynamically optimal to stop.

Theorem 2.1. *(i) Let $h : \mathbb{R}^+ \times S \to S$ be a deterministic function. If $X_t = h(t, x)$ $\mathbb{P}_x$-a.s, for any $x \in S$ and $t \geq 0$, then the minimal dynamically optimal stopping time in (2.1) is $\tau^\star = 0$.*

(ii) Let X be non-deterministic. Then, there is not a dynamically optimal stopping time $\tau^\star$ satisfying (2.1) such that $\mathbb{P}_x(\tau^\star < \infty) > 0$, for all $x \in S$.

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