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# Some families of asymmetric nested orthogonal arrays and asymmetric sliced orthogonal arrays

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ABSTRACT

This paper proposes construction algorithms for asymmetric nested and sliced orthogonal arrays with any strength. Based on group projection and algebraic techniques, families of asymmetric nested and sliced orthogonal arrays have been obtained. Examples are provided to illustrate these results.



## 1. Introduction

With the rapid development of computer technology, some physical experiments are simulated with complex computer programs. Space-filling designs are desirable for conducting computer experiments. Nested orthogonal arrays (NOAs) are useful in obtaining space-filling designs when an experimental situation consists of two experiments, the expensive one of higher accuracy (HE) to be nested in a larger and relatively inexpensive one of lower accuracy (LE). It is intuitively appealing to use nested space-filling designs to collect data from a pair of HE and LE for the purpose of building an accurate prediction model for the HE, see Qian et al. (2014) and Zhao and Zhao (2015).

Symmetric NOAs have been analyzed by Mukerjee et al. (2008), Qian et al. (2009a, b) and Sun et al. (2014). For asymmetric cases, Qian et al. (2014) showed the nested lattice samples were guaranteed to achieve uniformity in two or three dimensions only. In this work, we extend the above results and construct asymmetric NOAs with any strength, which generate asymmetric nested lattice samples with better space-filling properties.

On the other hand, Qian and Wu (2009) proposed sliced orthogonal arrays (SOAs) for space-filling designs with both qualitative and quantitative factors. When collapsed over the qualitative levels, the points of quantitative factors achieve attractive stratification. Recently, Ai et al. (2014) constructed sliced space-filling designs for symmetric SOAs. Li et al. (2015) considered asymmetric balanced sliced orthogonal arrays. However, their asymmetric results only achieved two dimensional uniformity. In order to overcome this limitation, we construct asymmetric sliced orthogonal arrays with any strength that can accommodate nesting with an arbitrary number of layers.

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The remainder is organized as follows. Some basic definitions and notations are reviewed in Section 2. Construction algorithms of asymmetric NOAs and SOAs are provided in Section 3. In Section 4, some families of asymmetric NOAs and SOAs are obtained and examples are given to illustrate the main results. A short conclusion is in Section 5.

## 2. Preliminaries and notations

An orthogonal array (OA) with $n$ runs, $m$ factors, $s_1, \dots, s_m$ symbols and strength $g$ $(1 \le g \le m)$, denoted by $OA(n, m, s_1 \times \cdots \times s_m, g)$, is an $n \times m$ matrix with symbols in the $i$th column from a finite set of $s_i \ge 2$ $(1 \le i \le m)$ symbols, such that in every $n \times g$ submatrix, all possible combinations of symbols appear equally often as a row. In particular, if $s_1 = \cdots = s_m = s$, then the array reduces to a symmetric OA, denoted simply by $OA(n, m, s, g)$. Otherwise, the array is *an asymmetric OA*.

From Mukerjee et al. (2008), we introduce the concept of NOAs. Let $A$ be an $OA(n, m, s_1 \times \cdots \times s_m, g)$ and $\rho_i$ be a series of projections for $i = 1, \dots, m$. Suppose $B \subset A$ and becomes an $OA(n_1, m, t_1 \times \cdots \times t_m, g)$ after the $s_i$ levels of the $s_i$-level factors are collapsed to $t_i$ levels according to some level-collapsing projection $\rho_i$ for $i = 1, \dots, m$. Then $B$ is nested in $A$, which is called a nested orthogonal array (NOA), denoted by $NOA(A, B)$, where $n_1 < n$, $t_i < s_i$. If $s_1 = \cdots = s_m = s$, $t_1 = \cdots = t_m = t$ and $t < s$, then one obtains *a symmetric NOA*. Otherwise, the array is *an asymmetric NOA*.

For the sliced orthogonal array, suppose that rows of $A$ are partitioned into $v$ subarrays of $n_2$ rows, denoted by $A_1, \dots, A_v$. Further, each $A_j$ $(j = 1, \dots, v)$ becomes an $OA(n_2, m, t_1 \times \cdots \times t_m, g)$ after the $s_i$ levels of the $s_i$-level factors are collapsed to $t_i$ levels according to some level-collapsing projection $\rho_i$ for $i = 1, \dots, m$. Then $A$, or more precisely $(A'_1, \dots, A'_v)'$, is called *a sliced orthogonal array (SOA)*. Provided that $s_i$'s are not all the same, it is *an asymmetric SOA*.

Now consider two matrices $A = (A_{ij}) = (a_1, \dots, a_s)$ with order $r \times s$ and $B = (b_{ij}) = (b_1, \dots, b_v)$ with order $u \times v$, respectively. The Kronecker sum of $A$ and $B$ is an $ru \times sv$ matrix, defined by $A \oplus B = (a_{ij}J + B)$, where $J$ is the $u \times v$ matrix of ones. Especially, if $s = v$, we introduce an operation $A \oplus_c B = (a_1 \oplus b_1, \dots, a_s \oplus b_s)$, called column-wise Kronecker sum of $A$ and $B$. The above operations will be used to construct asymmetric NOAs and asymmetric SOAs in the next sections.

Following the terminology in factorial experiments, it is convenient to call the columns of an arbitrary $OA(n, m, s_1 \times \cdots \times s_m, g)$ factors, denoted by $F_1, \dots, F_m$. Let $GF(s)$ be a Galois field of order $s$ with 0 and 1 denoting the identity elements of $GF(s)$ with respect to the operations of addition and multiplication. For the factor $F_i$ $(1 \le i \le m)$, define $u_i$ columns, say $\mathbf{p}_{i_1}, \dots, \mathbf{p}_{i_{u_i}}$, each of order $k \times 1$ with elements from $GF(s)$. Thus, for the $m$ factors, we have $\sum_{i=1}^{m} u_i$ columns in all. Let $H$ be an $s^k \times k$ matrix whose rows are all possible $k$-tuple over $GF(s)$ and $P_i = (\mathbf{p}_{i_1}, \dots, \mathbf{p}_{i_{u_i}})$ for $1 \le i \le m$. Suen et al. (2001) provided a method to construct $OA(s^k, m, (s^{u_1}) \times (s^{u_2}) \times \cdots \times (s^{u_m}), g)$.

**Lemma 1.** *Consider a $k \times \sum_{i=1}^{m} u_i$ matrix $C = (P_1, P_2, \dots, P_m)$ such that for every choice of $g$ matrices $P_{i_1}, \dots, P_{i_g}$ from $P_1, \dots, P_m$, the $k \times \sum_{j=1}^{g} u_{i_j}$ matrix $(P_{i_1}, \dots, P_{i_g})$ has full column rank over $GF(s)$. Then an $OA(s^k, m, (s^{u_1}) \times (s^{u_2}) \times \cdots \times (s^{u_m}), g)$ is constructed.*

Lemma 1 showed the above orthogonal array is constructed by $HC$ with the operations of addition and multiplication over $GF(s)$, where the $u_i$ columns of $F_i$ form a new column of $s^{u_i}$ symbols for $1 \le i \le m$. Based on the OA, we provide some construction algorithms for NOAs and SOAs in the next section.

## 3. Construction algorithms of asymmetric NOAs and SOAs

Sun et al. (2014) presented the subgroup projection and other algebraic techniques for constructing symmetric NOAs and SOAs. In this section, we extend the methods to construct asymmetric NOAs and SOAs. For convenience, we only consider asymmetric cases with two layers, which can be easily generalized to general layers. For a finite set $A$ of size $|A|$, put its elements in a column vector $V_A$ with zero being placed as the first entry if included. Based on Lemma 3 of Sun et al. (2014), the following lemma provides a way for decomposition of Galois field.

**Lemma 2.** *Suppose $T$ is a Galois field $GF(p^\lambda)$, where $p$ is a prime and $\lambda$ is a positive integer. If $T_1$ is a subgroup of $T$ under operation "+", then there exists a subgroup $T_2$ of $T$ under operation "+" satisfying $V_T = V_{T_1} \oplus V_{T_2}$.*

By Lemma 2, any $\gamma \in T$ is uniquely expressed as $\gamma = \beta_1 + \beta_2$, where $\beta_i \in T_i$ for $i = 1, 2$. Define a projection $\rho: T \to T_1$ as $\rho(\gamma) = \rho(\beta_1 + \beta_2) = \beta_1$, called *a subgroup projection*. For the subgroup projection $\rho$ and $\gamma_1, \gamma_2 \in T$, we have

$$\rho(\gamma_1 + \gamma_2) = \rho(\gamma_1) + \rho(\gamma_2). \tag{1}$$

In order to construct symmetrical NOAs, Sun et al. (2014) provided properties of the projection $\rho$. It is easy to be extended to asymmetrical cases.

**Lemma 3.** *Suppose $s_i = p^{\lambda_i}$ for $i = 1, 2$, where $p$ is a prime and $1 \le \lambda_2 < \lambda_1$. If $A$ is an $OA(s_1^k, m, (s_1^{u_1}) \times \cdots \times (s_1^{u_m}), g)$ based on $GF(s_1)$, $\rho$ is the subgroup projection from $GF(s_1)$ to $GF(s_2)$, then $\rho(A)$ is an $OA(s_1^k, m, (s_2^{u_1}) \times \cdots \times (s_2^{u_m}), g)$ based on $GF(s_2)$.*

Based on Lemmas 1–3, we introduce the following algorithm, which is used to construct both of asymmetrical NOAs and SOAs.

**Construction algorithm:**

Let $T = GF(s_1)$ with $s_1 = p^{\lambda_1}$ elements, $T_1$ has $s_2 = p^{\lambda_2}$ elements and is a subgroup of $T$ under operation "+", where $1 \le \lambda_2 < \lambda_1$. There are four steps for constructing asymmetrical NOAs and SOAs as follows: