Statistics and Probability Letters 140 (2018) 71–76

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Moderate deviations for multivariate Hawkes processes

Nian Yao

School of Mathematics and Statistics, Shenzhen University, 518060 Shenzhen, China

ARTICLE INFO

Article history:
Received 8 February 2018
Received in revised form 21 April 2018
Accepted 22 April 2018
Available online 4 May 2018

MSC:
60G55
60F10

Keywords:
Moderate deviations
Large deviations
Multivariate Hawkes processes
Point processes
Self-exciting processes

ABSTRACT

The Hawkes process is a self-exciting simple point process. In this paper, we study the Moderate Deviation Principle for multivariate Hawkes processes on the large time regimes.

1. Introduction

1.1. Motivation and setting

We call a multivariate d-dimensional counting process $N := (N^1, \dots, N^d)$ with values in $\mathbb{N}^d$ a multivariate Hawkes process if N^i is a simple point process with intensity

$$\lambda_{i,t} = \nu_i + \sum_{j=1}^{d} \int_{(0,t)} h_{ij}(t-s)N^j(ds), \quad 1 \le i \le d, \tag{1}$$

where $\nu_i > 0$, $1 \le i \le d$ are baseline intensities and $h_{ij}(\cdot) : \mathbb{R}^+ \to [0, \infty)$, $1 \le i, j \le d$ are exciting functions. We assume that $N = (N^1, \dots, N^d)$ has empty history, i.e. $N^i(-\infty, 0] = 0$ for any $1 \le i \le d$.

A temporal point process is a stochastic process which plays an important role in the analysis of observed patterns of points, where the points represent the locations of some object of study. Hawkes process is perhaps the most parsimonious self-exciting point process whose conditional intensity function is linear and increasing. The linear Hawkes process was first introduced by Hawkes (1971a, b). It naturally generalizes the Poisson process and is able to capture both the self-exciting property and the clustering effect. Hawkes process is a very versatile model which is amenable to statistical analysis, being evident by its wide applications in neuroscience, genome analysis, criminology, social networks, seismology, insurance, finance and many other fields, for example, see Zhu (2013b) and references therein. The moderate deviation principle for univariate linear Hawkes process was discussed in Zhu (2013a) (unmarked case) and Seol (2017) (marked case). For multivariate Hawkes processes, Bacry et al. (2013) proved the functional law of large numbers and the functional central

E-mail address: yaonian@szu.edu.cn.

https://doi.org/10.1016/j.spl.2018.04.023

limit theorems. Delattre et al. (2016) for the first time studied mean-field limit. Chevallier (2017) obtained the mean-field limit of generalized multivariate Hawkes processes and Chevallier et al. (2017) for nonlinear spatially extended Hawkes processes with exponential memory kernels. The fluctuations, moderate deviations and large deviations for the mean process for multivariate Hawkes processes are studied in Gao and Zhu (2018a).

In this paper, we focus on the study of moderate deviations for multivariate Hawkes processes. The study of moderate deviations has been an active research area, and over the years certain mixing processes, Markov processes, martingales, etc., have been studied. It is well known for moderate deviation for i.i.d. random variables. However, it is in general difficult to obtain moderate deviations for dependent random variables.

The main contributions of this paper are given by following:

1. Based on the immigration–birth representation of Hawkes processes, we obtain the characterization of the moment generating function of multivariate Hawkes processes (Lemma 5);
2. We establish the corresponding moderate deviation principle for multivariate Hawkes processes (Theorem 2)

In particular, we generalize the moderate deviations for univariate linear Hawkes process appeared in Zhu (2013a) to the ones of multivariate Hawkes processes. From the application point of view, multivariate Hawkes processes are the most realistic model to characterize the self-exciting and mutual-exciting properties, clustering effect and the sensitivity of the intensity. Moreover, multivariate Hawkes processes have already been widely used to model the complex systems like stock indices movement, neuronal networks etc.

1.2. Statement of main results

We start with the assumptions which we will use throughout the paper.

Assumption 1. Let $K_{ij} := \int_0^\infty h_{ij}(t)dt$ and $K := (K_{ij})_{1\le i,j\le d}$.
(A1) $K_{ij} < \infty$ for $1 \le i, j \le d$ and $\rho(K) < 1$ where $\rho(K)$ is the spectral radius of K.
(A2) $\max_{1\le i,j\le d}\sup_{t>0} t^{3/2} h_{ij}(t) = C < \infty$.

Bacry et al. (2013) proved the functional law of large numbers and the functional central limit theorems for multivariate Hawkes processes, and in particular we also have the scalar law of large numbers:

$$\frac{N_t}{t} \to \mu := (I-K)^{-1}\nu, \tag{2}$$

where $\nu = (\nu_1, \dots, \nu_d)$ and under the additional assumption $\int_0^\infty t^{1/2} h_{ij}(t)dt < \infty$, the scalar central limit theorem:

$$\frac{N_t - \mu t}{\sqrt{t}} \to (I-K)^{-1}\Sigma^{1/2}N(0, I), \tag{3}$$

in distribution as $t \to \infty$, where Σ is the diagonal matrix such that $\Sigma_{ii} = ((I-K)^{-1}\nu)_i$ for $1 \le i \le d$.

We obtain in this section a moderate deviation principle for linear multivariate Hawkes process. We start with the basic definitions in a moderate deviations theory (e.g. see Dembo and Zeitouni (1998)). Let $X_1, \dots, X_n$ be a sequence of $\mathbb{R}^d$-valued i.i.d. random vectors with mean 0 and covariance matrix C which is invertible. Assume that $\mathbb{E}[e^{\langle\theta, X_1\rangle}] < \infty$ for θ in some ball around the origin. For any $\sqrt{n} \ll a_n \ll n$, a moderate deviation principle says that, for any Borel set A in $\mathbb{R}^d$,

$$\begin{aligned} -\frac{1}{2}\inf_{x\in A^\circ}\langle x, C^{-1}x\rangle &\le \liminf_{n\to\infty}\frac{n}{a_n^2}\log\mathbb{P}\left(\frac{1}{a_n}\sum_{i=1}^n X_i \in A\right) \\ &\le \limsup_{n\to\infty}\frac{n}{a_n^2}\log\mathbb{P}\left(\frac{1}{a_n}\sum_{i=1}^n X_i \in A\right) \le -\frac{1}{2}\inf_{x\in \bar{A}}\langle x, C^{-1}x\rangle. \end{aligned} \tag{4}$$

In other words, $\mathbb{P}(\frac{1}{a_n}\sum_{i=1}^n X_i \in \cdot)$ satisfies a large deviation principle with speed $\frac{a_n^2}{n}$. The moderate deviation principle fills in the gap between the central limit theorem (CLT) and the large deviation principle (LDP) for the models.

Theorem 2. *For any Borel set A in $\mathbb{R}^d$ and time sequence $a(t)$ such that $\sqrt{t} \ll a(t) \ll t$, we have the following moderate deviation principle*

$$\begin{aligned} -\inf_{x\in A^\circ} J(x) &\le \liminf_{t\to\infty}\frac{t}{a(t)^2}\log\mathbb{P}\left(\frac{N_t-\mu t}{a(t)} \in A\right) \\ &\le \limsup_{t\to\infty}\frac{t}{a(t)^2}\log\mathbb{P}\left(\frac{N_t-\mu t}{a(t)} \in A\right) \le -\inf_{x\in\bar{A}} J(x) \end{aligned} \tag{5}$$

where $J(x) = \frac{1}{2}x^T(I-K^T)\Sigma^{-1}(I-K)x$.

Remark 3. We can easily check the corresponding $C = (I-K)^{-1}\Sigma(I-K^T)^{-1}$ for $J(x) = \frac{1}{2}\langle x, C^{-1}x\rangle$ in (26). That is just the covariance matrix in Bacry et al. (2013)'s functional central limit theorem.