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Dual representations of Laplace transforms of Brownian excursion and generalized meanders

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ABSTRACT

The Laplace transform of the d-dimensional distribution of Brownian excursion is expressed as the Laplace transform of the $(d+1)$-dimensional distribution of an auxiliary Markov process, started from a σ-finite measure and with the roles of arguments and times interchanged. A similar identity holds for the Laplace transform of a generalized Brownian meander, which is expressed as the Laplace transform of the same auxiliary Markov process, with a different initial law.

1. Introduction

Consider the Brownian excursion standardized to have length 1 and conditioned to be positive. This is also the Brownian bridge conditioned to stay positive, or the 3-dimensional Bessel process conditioned to hit zero at $t = 1$, and its finite-dimensional distributions are given by formula (2.1). We let $\mathbb{B}^{ex} = (\mathbb{B}^{ex}_t)_{t\in[0,1]}$ denote this process throughout. Brownian excursion has been extensively investigated in the literature. See for example Bertoin and Pitman (1994) and Revuz and Yor (1999). It also appears in asymptotic analysis of various combinatorial problems, see for example Pitman (2006) and Janson (2007).

The purpose of this note is to introduce a "dual representation" that ties the Laplace transforms of finite-dimensional distributions of Brownian excursion and another Markov process, denoted by $(X_t)_{t\geq 0}$ throughout, with state space $[0,\infty)$ and transition probabilities

$$\mathbb{P}(X_t \in dy \mid X_s = x) = p_{t-s}(x,y)dy,\ \ 0 \leq s < t, x \geq 0$$

with

$$p_t(x,y) = \frac{2t\sqrt{y}}{\pi\left[(y-x)^2 + 2(x+y)t^2 + t^4\right]},\ \ t > 0, y \geq 0. \tag{1.1}$$

This is a positive self-similar Markov process that arises as the tangent process at the boundary of support of so-called q-Brownian motions and q-Ornstein–Uhlenbeck processes; see Bryc and Wang (2016, Proposition 2.2) and Wang (2017, Theorem 3.1). It can also be obtained from the construction in Biane (1998) applied to the 1/2-stable free Lévy process by

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including appropriate drift. The derivation of the transition probability density function, following Biane's approach, can be found in Bryc and Wang (2016, Section 3).

Our main result is the following identity. Let $\mathbb{E}_x[\cdot]$ denote the expectation with respect to the law of $(X_t)_{t\geq 0}$ starting at $X_0 = x > 0$.

Theorem 1.1. *For $d \in \mathbb{N}$, let $s_0 = 0 < s_1 < s_2 < \cdots < s_d$ and $t_0 = 0 \leq t_1 < \cdots < t_d \leq 1 = t_{d+1}$. Then,*

$$\mathbb{E}\left[\exp\left(-\sum_{k=1}^{d}(s_k - s_{k-1})\mathbb{B}^{ex}_{t_k}\right)\right] = \frac{1}{\sqrt{2\pi}}\int_0^\infty \mathbb{E}_x\left[\exp\left(-\frac{1}{2}\sum_{k=0}^{d}(t_{k+1}-t_k)X_{s_k}\right)\right]\sqrt{x}\,dx. \tag{1.2}$$

The left-hand side of (1.2) is the Laplace transform of the joint distribution of Brownian excursion, while the right-hand side is the Laplace transform of the process $(X_t)_{t\geq 0}$ with the arguments and time indices interchanged. On the right-hand side, the initial distribution of $(X_t)_{t\geq 0}$ is the stationary σ-finite measure $(x/(2\pi))^{1/2}\mathbf{1}_{\{x>0\}}dx$.

We are aware of only a couple of results that connect Laplace transforms of stochastic processes by interchanging the argument and time parameters. One such result is the formula for the joint generating function of the finite Asymmetric Simple Exclusion Process in Bryc and Wesołowski (2017, Theorem 1). Another result of this type is the formula Bertoin and Yor (2001, Eq. (2)) for the univariate Mellin transform of a positive self-similar Markov process, see also Hirsch and Yor (2012).

As an immediate consequence of Theorem 1.1 however, a family of such dualities can be derived easily for *generalized Brownian meanders* defined in (3.1), including some of the generalized Bessel meanders considered in Mansuy and Yor (2008, Section 3.7). The dualities, see Theorem 3.1 and Corollary 3.3, have similar forms as in (1.2), but they differ in the choice of the σ-finite measure for the initial law on the right-hand side of (1.2).

Due to its importance, here we state the duality formula for Brownian meander $(\mathbb{B}^{me}_t)_{0\leq t\leq 1}$, which is a special case $\delta = 1$ of Corollary 3.3. Recall that Brownian meander can be defined as the Brownian motion conditioned to stay positive over the time interval $[0, 1]$. See for example Bertoin and Pitman (1994) and Pitman (1999, 2006). We also need this formula for our analysis in Bryc and Wang (2017) for asymmetric simple exclusion processes.

Theorem 1.2. *For $d \in \mathbb{N}$, let $s_0 = 0 < s_1 < s_2 < \cdots < s_d$ and $t_0 = 0 \leq t_1 < t_2 < \cdots < t_d \leq 1 = t_{d+1}$. Then*

$$\mathbb{E}\left[\exp\left(-\sum_{k=1}^{d}(s_k - s_{k-1})\mathbb{B}^{me}_{1-t_k}\right)\right] = \frac{1}{\sqrt{2\pi}}\int_0^\infty \mathbb{E}_x\left[\exp\left(-\frac{1}{2}\sum_{k=0}^{d}(t_{k+1}-t_k)X_{s_k}\right)\right]\frac{1}{\sqrt{x}}\,dx. \tag{1.3}$$

Formulas (1.2) and (1.3) were needed in Bryc and Wang (2017), where we investigated, by essentially computing the Laplace transforms, the fluctuations of asymmetric simple exclusion processes with open boundaries in the steady state, and were first discovered in that setting with similar proofs. However, it turns out that formula (1.3) can be derived directly from (1.2) as a special case of an entire family of formulas that appear in Theorem 3.1, so we decided to present these results separately.

The paper is organized as follows. In Section 2 we recall some known facts about Brownian excursion and prove Theorem 1.1. In Section 3 we prove the dual representations for generalized Brownian meanders, with Brownian meander as a special case.

2. Proof of dual representation of Brownian excursion

We first recall some facts on Brownian excursion $\mathbb{B}^{ex}$. For our purposes it is convenient to define it as a Markov process that starts at $\mathbb{B}^{ex}_0 = 0$, ends at $\mathbb{B}^{ex}_1 = 0$, and has transition probabilities

$$P(\mathbb{B}^{ex}_t \in dy \mid \mathbb{B}^{ex}_s = x) = \begin{cases} \sqrt{8\pi}\,\ell_t(y)\ell_{1-t}(y) & \text{if } s = 0 < t < 1, x = 0, y > 0 \\ g_{t-s}(x,y)\dfrac{\ell_{1-t}(y)}{\ell_{1-s}(x)} & \text{if } 0 < s < t < 1, x > 0, y > 0, \end{cases}$$

with

$$\ell_t(y) = \frac{1}{\sqrt{2\pi t^3}} \cdot y\exp\left(-\frac{y^2}{2t}\right)\mathbf{1}_{\{y>0\}}$$