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Some new results on Triple designs

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ABSTRACT

The method of doubling has been used to construct two-level fractional factorial designs. Zhang (2016) extended the method of doubling to the method of tripling, which has been used to construct three-level fractional factorial designs. Various popular screening criteria, such as $E(f_{NOD})$ criterion, generalized minimum aberration (GMA), minimum moment aberration (MMA), B criterion and uniformity criterion, have been proposed from different viewpoints to compare and construct designs. In this paper, the analytic connections between Triple designs constructed by tripling and initial designs under the above screening criteria are considered. These connections are suitable to general original three-level fractional factorial designs, whether regular or nonregular. Furthermore, the connections provide theoretical basis for constructing a kind of optimal three-level designs with large sizes by tripling successively.

1. Introduction

The method of doubling has been used to construct two-level fractional factorial designs. Suppose X is a two-level design with n runs and s factors, which can be presented as an $n \times s$ matrix with two distinct entries, $+1$ and -1. The $2n \times 2s$ matrix $\begin{pmatrix} X & X \\ X & -X \end{pmatrix}$ is called the doubling of X, denoted by $D(X)$, it is obvious that $D(X)$ is also a two-level design and doubles both the run size and the number of factors of X. $D(X)$ is called the Double design of X, X is called the initial design of $D(X)$. Chen and Cheng (2006) discussed to construct $D(X)$ of resolution IV via a regular initial design X of resolution IV, and proved that there exists a projection design of $D(X)$ with resolution IV or higher. Xu and Cheng (2008) studied a general complementary design theory for doubling, and the connection of wordlength patterns between each pair of complementary projection designs are built by repeated doubling. Ou and Qin (2010) considered some links of the indicator function between $D(X)$ and X. Lei and Qin (2014) studied the uniformity measured by centered L_2-discrepancy of $D(X)$, some lower bounds of centered L_2-discrepancy value for $D(X)$ are obtained, in addition, the connection of uniformity between Double designs and initial designs are also investigated.

Zhang (2016) extended the method of doubling to the method of tripling based on level permutation. Suppose $\mathcal{A}$ is a three-level design with n runs and m factors, which can be presented as an $n \times m$ matrix with three distinct entries, 0, 1 and 2. Table 1 lists six designs obtained from all possible level permutations of $\mathcal{A}$. The $3n \times 3m$ matrix $\begin{pmatrix} \mathcal{A} & \mathcal{A} & \mathcal{A}_{(1)} \\ \mathcal{A} & \mathcal{A}_{(4)} & \mathcal{A}_{(2)} \\ \mathcal{A} & \mathcal{A}_{(5)} & \mathcal{A}_{(3)} \end{pmatrix}$,

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Table 1
Six designs via level permutations of $\mathcal{A}$.

Permutation No.	Initial design	Permutation method	Image
1	$\mathcal{A}$	$(0, 1, 2) \mapsto (0, 1, 2)$	$\mathcal{A}$
2	$\mathcal{A}$	$(0, 1, 2) \mapsto (0, 2, 1)$	$\mathcal{A}_{(1)}$
3	$\mathcal{A}$	$(0, 1, 2) \mapsto (2, 1, 0)$	$\mathcal{A}_{(2)}$
4	$\mathcal{A}$	$(0, 1, 2) \mapsto (1, 0, 2)$	$\mathcal{A}_{(3)}$
5	$\mathcal{A}$	$(0, 1, 2) \mapsto (2, 0, 1)$	$\mathcal{A}_{(4)}$
6	$\mathcal{A}$	$(0, 1, 2) \mapsto (1, 2, 0)$	$\mathcal{A}_{(5)}$

is called the tripling of $\mathcal{A}$, denoted by $\mathcal{T}(\mathcal{A})$, where $\mathcal{A}_{(i)}, i = 1, 2, \ldots, 5$, is listed in Table 1. It is obvious that $\mathcal{T}(\mathcal{A})$ is also a three-level design and triples both the run size and the number of factors of $\mathcal{A}$. $\mathcal{T}(\mathcal{A})$ is called the Triple design of $\mathcal{A}$, $\mathcal{A}$ is called the initial design of $\mathcal{T}(\mathcal{A})$. Zhang (2016) considered some links of the indicator functions between the Triple design $\mathcal{T}(\mathcal{A})$ and its original design $\mathcal{A}$, the uniformity of $\mathcal{T}(\mathcal{A})$ and projective designs of $\mathcal{T}(\mathcal{A})$ was also studied.

Tripling is an extension of doubling. A natural question arises: what are connections between Triple designs and original designs under various screening criteria? The present paper aims to study the question and to provide theoretical basis for constructing optimal three-level designs with large sizes by tripling successively. The paper is organized as follows. In Section 2, various screening criteria are reviewed and an important lemma is provided. Analytic connections between Triple designs and initial designs under the different criteria are investigated and some properties are obtained in Section 3, moreover, two illustrative examples are also given in this section to support our theoretical results. Some concluding remarks are given in Section 4.

2. Preliminaries

Consider a class of designs, denoted as $\mathcal{D}(n; 3^m)$, of a three-level fractional factorial design with n runs and m factors, each factor has three levels, say, $\{0, 1, 2\}$. Any three-level factorial design $\mathcal{A} \in \mathcal{D}(n; 3^m)$ can be represented by an $n \times m$ matrix $(x_{ij})_{n\times m}$ with entries 0, 1, 2, where each row represents a run (or level combination), and each column represents a factor. Next, some popular criteria for comparing factorial designs are briefly described.

For a three-level factorial design $\mathcal{A} \in \mathcal{D}(n; 3^m)$, When $2m = n - 1$, the design $\mathcal{A}$ is called saturated. When $2m > n - 1$, the design $\mathcal{A}$ is called supersaturated. Let x^k and x^l be kth and lth columns of the design $\mathcal{A}$, n^{kl}_{uv} is the number of (u, v)-pairs in (x^k, x^l), $f^{kl}_{NOD} = \sum_{u=0}^{2}\sum_{v=0}^{2}\left(n^{kl}_{uv} - \frac{n}{9}\right)^2$. The $E(f_{NOD})$ criterion, proposed by Fang et al. (2003), minimizes $E(f_{NOD}) = \sum_{1\le k<l\le m} f^{kl}_{NOD}/\binom{m}{2}$, which measures two-factor non-orthogonality combinatorially. Let $\lambda_{ij}(\mathcal{A})$ be the number of coincidences between the two rows x_i and x_j of $\mathcal{A}$, Fang et al. (2003) provided an analytic connection between $E(f_{NOD})$ and $\lambda_{ij}(\mathcal{A})$, and obtained a lower bound of $E(f_{NOD})$ via $\lambda_{ij}(\mathcal{A})$, that is,

$$E(f_{NOD}) = \frac{1}{m(m-1)} \sum_{i,j(i\neq j)=1}^{n} [\lambda_{ij}(\mathcal{A})]^2 + \frac{n}{m-1}\left(m - \frac{n}{3}\right) - \frac{n^2}{9}, \quad (1)$$

and

$$E(f_{NOD}) \geq LE(f_{NOD}), \quad (2)$$

where $LE(f_{NOD}) = \frac{mn}{(n-1)(m-1)}\left(\frac{n}{3} - 1\right)^2 + \frac{n}{m-1}\left(m - \frac{n}{3}\right) - \left(\frac{n}{3}\right)^2$. The equality holds in (2) if and only if all the $\lambda_{ij}(\mathcal{A})$s are equal to λ for $i \neq j$, where $\lambda = m(n/3 - 1)/(n - 1)$ is a positive integer.

For any design $\mathcal{A} \in \mathcal{D}(n; 3^m)$, the distance distribution $(E_0(\mathcal{A}), \ldots, E_m(\mathcal{A}))$ of $\mathcal{A}$ is defined as

$$E_i(\mathcal{A}) = \frac{1}{n}|\{(a, b) : d_H(a, b) = i, a \text{ and } b \text{ are two runs of } \mathcal{A}\}|, \quad 0 \le i \le m, \quad (3)$$

where $d_H(a, b)$ is the Hamming distance between two rows a and b, that is, the number of places where they differ, $|\Omega|$ is the cardinality of Ω. Based on the distance distribution $(E_0(\mathcal{A}), \ldots, E_m(\mathcal{A}))$ of design $\mathcal{A}$, the generalized wordlength pattern $(A_1(\mathcal{A}), \ldots, A_m(\mathcal{A}))$ of $\mathcal{A}$ is defined as

$$A_j(\mathcal{A}) = \frac{1}{n}\sum_{i=0}^{m} P_j(i; m, 3)E_i(\mathcal{A}), \quad j = 1, \ldots, m, \quad (4)$$

where $P_j(i; m, 3) = \sum_{r=0}^{j}(-1)^r 2^{j-r}\binom{i}{r}\binom{m-i}{j-r}$ is Krawtchouk polynomial, $\binom{x}{y} = x(x-1)\cdots(x-y+1)/y!$, $\binom{x}{0} = 1$, for $x < y$, $\binom{x}{y} = 0$. The GMA criterion is to sequentially minimize $A_j(\mathcal{A})$ for $j = 1, \ldots, m$. One can refer to Xu and Wu (2001) for more details.

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