Statistics and Probability Letters 140 (2018) 106–114

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Regularly varying non-stationary Galton–Watson processes with immigration

Mátyás Barczy [a,*], Zsuzsanna Bősze [b], Gyula Pap [b]

[a] MTA-SZTE Analysis and Stochastics Research Group, Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, H–6720 Szeged, Hungary

[b] Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, H-6720 Szeged, Hungary

ARTICLE INFO

Article history:
Received 11 January 2018
Received in revised form 7 April 2018
Accepted 3 May 2018
Available online 12 May 2018

MSC:
60J80
60G70

Keywords:
Galton–Watson process with immigration
Regularly varying distribution

ABSTRACT

We give sufficient conditions on the initial, offspring and immigration distributions under which the distribution of a not necessarily stationary Galton–Watson process with immigration is regularly varying at any fixed time.

1. Introduction

Galton–Watson processes with immigration have been frequently used for modeling the sizes of a population over time, so a delicate description of their tail behavior is an important question. In this paper we focus on regularly varying not necessarily stationary Galton–Watson processes with immigration, complementing the results of Basrak et al. (2013) for the stationary case. By a Galton–Watson process with immigration, we mean a stochastic process $(X_n)_{n\geqslant 0}$ given by

$$X_n = \sum_{i=1}^{X_{n-1}} \xi_{n,i} + \varepsilon_n, \qquad n \geqslant 1, \tag{1.1}$$

where $\{X_0,\ \xi_{n,i},\ \varepsilon_n : n, i \geqslant 1\}$ are supposed to be independent non-negative integer-valued random variables, $\{\xi_{n,i} : n, i \geqslant 1\}$ and $\{\varepsilon_n : n \geqslant 1\}$ are supposed to consist of identically distributed random variables, respectively, and $\sum_{i=1}^{0} := 0$. If $\varepsilon_n = 0$, $n \geqslant 1$, then we say that $(X_n)_{n\geqslant 0}$ is a Galton–Watson process (without immigration).

Basrak et al. (2013) studied stationary Galton–Watson processes with immigration and gave conditions under which the stationary distribution is regularly varying.

In the special case of $\mathbb{P}(\xi_{1,1} = \varrho) = 1$ with some non-negative integer ϱ, $(X_n)_{n\geqslant 0}$ is nothing else but a first order autoregressive process having the form $X_n = \varrho X_{n-1} + \varepsilon_n$, $n \geqslant 1$. There is a vast literature on the tail behavior of weighted sums of independent and identically distributed regularly varying random variables, especially, of first order autoregressive processes with regularly varying noises, see, e.g., Embrechts et al. (1997, Appendix A3.3). For instance, in the special case

* Corresponding author.
E-mail addresses: barczy@math.u-szeged.hu (M. Barczy), Bosze.Zsuzsanna@stud.u-szeged.hu (Z. Bősze), papgy@math.u-szeged.hu (G. Pap).

https://doi.org/10.1016/j.spl.2018.05.010
0167-7152/

mentioned before, Proposition 2.4 with $\mathbb{P}(X_0 = 0) = 1$ gives the result of Lemma A3.26 in Embrechts et al. (1997), since then $X_n = \sum_{i=1}^{n} \varrho^{n-i}\varepsilon_i$, $n \geqslant 1$.

In Section 2, we present conditions on the initial, offspring and immigration distributions under which the distribution of a not necessarily stationary Galton–Watson process with immigration is regularly varying at any fixed time describing the precise tail behavior of the distribution in question as well. The proofs are delicate applications of Faÿ et al. (2006, Proposition 4.3) (see Proposition D.1), Robert and Segers (2008, Theorem 3.2) (see Proposition D.2) and Denisov et al. (2010, Theorems 1 and 7) (see Propositions D.4 and D.6). We close the paper with four Appendices: in Appendix A we recall representations of Galton–Watson process without or with immigration; Appendix B is devoted to higher moments of Galton–Watson processes; in Appendix C we collect some properties of regularly varying functions and distributions used in the paper; and in Appendix D we recall the results of Faÿ et al. (2006, Proposition 4.3), Robert and Segers (2008, Theorem 3.2), Denisov et al. (2010, Theorems 1 and 7) and some of their consequences.

Later on, one may also investigate other tail properties such as intermediate regular variation. Motivated by Bloznelis (2018), one may study the asymptotic behavior of the so called local probabilities $\mathbb{P}(X_n = \ell)$ as $\ell \to \infty$ for any fixed $n \in \mathbb{N}$.

2. Tail behavior of Galton–Watson processes with immigration

Let $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the set of non-negative integers, positive integers, real numbers, non-negative real numbers and positive real numbers, respectively. For $x, y \in \mathbb{R}$, we will use the notations $x \wedge y := \min(x, y)$ and $x \vee y := \max(x, y)$. For functions $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$ and $g : \mathbb{R}_{++} \to \mathbb{R}_{++}$, by the notation $f(x) \sim g(x)$, $f(x) = \mathrm{o}(g(x))$ and $f(x) = \mathrm{O}(g(x))$ as $x \to \infty$, we mean that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ and $\limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty$, respectively. Every random variable will be defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Equality in distribution of random variables is denoted by $\stackrel{\mathcal{D}}{=}$. For notational convenience, let ξ and ε be random variables such that $\xi \stackrel{\mathcal{D}}{=} \xi_{1,1}$ and $\varepsilon \stackrel{\mathcal{D}}{=} \varepsilon_1$, and put $m_\xi := \mathbb{E}(\xi) \in [0, \infty]$ and $m_\varepsilon := \mathbb{E}(\varepsilon) \in [0, \infty]$.

First, we consider the case of regularly varying offspring distribution.

Proposition 2.1. *Let $(X_n)_{n\in\mathbb{Z}_+}$ be a Galton–Watson process with immigration such that ξ is regularly varying with index $\alpha \in [1, \infty)$ and there exists $r \in (\alpha, \infty)$ with $\mathbb{E}(X_0^r) < \infty$ and $\mathbb{E}(\varepsilon^r) < \infty$. Suppose that $\mathbb{P}(X_0 = 0) < 1$ or $\mathbb{P}(\varepsilon = 0) < 1$. In case of $\alpha = 1$, assume additionally that $m_\xi \in \mathbb{R}_{++}$. Then for each $n \in \mathbb{N}$, we have*

$$\mathbb{P}(X_n > x) \sim \mathbb{E}(X_0) m_\xi^{n-1} \sum_{i=0}^{n-1} m_\xi^{i(\alpha-1)} \mathbb{P}(\xi > x) + m_\varepsilon \sum_{i=1}^{n-1} m_\xi^{n-i-1} \sum_{j=0}^{n-i-1} m_\xi^{j(\alpha-1)} \mathbb{P}(\xi > x) \qquad \text{as } x \to \infty,$$

and hence X_n is also regularly varying with index α.

Proof. Note that we always have $\mathbb{E}(X_0) \in \mathbb{R}_+$, $m_\xi \in \mathbb{R}_{++}$ and $m_\varepsilon \in \mathbb{R}_+$. We use the representation (A.2). Recall that $\left\{V^{(n)}(X_0), V_i^{(n-i)}(\varepsilon_i) : i \in \{1, \ldots, n\}\right\}$ are independent random variables such that $V^{(n)}(X_0)$ represents the number of individuals alive at time n, resulting from the initial individuals X_0 at time 0, and for each $i \in \{1, \ldots, n\}$, $V_i^{(n-i)}(\varepsilon_i)$ represents the number of individuals alive at time n, resulting from the immigration ε_i at time i. If $\mathbb{P}(X_0 = 0) = 1$, then $\mathbb{P}(V^{(n)}(X_0) = 0) = 1$, otherwise, by Proposition D.5, we obtain

$$\mathbb{P}(V^{(n)}(X_0) > x) \sim \mathbb{E}(X_0) m_\xi^{n-1} \sum_{i=0}^{n-1} m_\xi^{i(\alpha-1)} \mathbb{P}(\xi > x) \qquad \text{as } x \to \infty$$

once we show

$$\mathbb{P}(V_n > x) \sim m_\xi^{n-1} \sum_{i=0}^{n-1} m_\xi^{i(\alpha-1)} \mathbb{P}(\xi > x) \qquad \text{as } x \to \infty, \tag{2.1}$$

where $(V_k)_{k\in\mathbb{Z}_+}$ is a Galton–Watson process (without immigration) with initial value $V_0 = 1$ and with the same offspring distribution as $(X_k)_{k\in\mathbb{Z}_+}$. We proceed by induction on n. For $n = 1$, (2.1) follows readily, since $V_1 = \xi_{1,1} \stackrel{\mathcal{D}}{=} \xi$. Now let us assume that (2.1) holds for $1, \ldots, n-1$, where $n \geqslant 2$. Since $(V_k)_{k\in\mathbb{Z}_+}$ is a time homogeneous Markov process with $V_1 = \xi_{1,1}$, we have $V_n \stackrel{\mathcal{D}}{=} V^{(n-1)}(\xi_{1,1})$, where $(V^{(k)}(\xi_{1,1}))_{k\in\mathbb{Z}_+}$ is a Galton–Watson process (without immigration) with initial value $V^{(0)}(\xi_{1,1}) = \xi_{1,1}$ and with the same offspring distribution as $(X_k)_{k\in\mathbb{Z}_+}$. Applying again the additive property (A.1), we obtain

$$V_n \stackrel{\mathcal{D}}{=} V^{(n-1)}(\xi_{1,1}) \stackrel{\mathcal{D}}{=} \sum_{i=1}^{\xi_{1,1}} \zeta_i^{(n-1)},$$