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The Lambert W function, Nuttall's integral, and the Lambert law

Anthony G. Pakes

Department of Mathematics & Statistics, University of Western Australia, 35 Stirling Highway, Crawley WA, 6009, Australia

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ABSTRACT

This paper offers a new proof that the principal Lambert W-function $W(s)$ is a Bernstein function. The proof derives from a known integral evaluation and leads to a more detailed description of $W(s)$ as a Thorin–Bernstein function with a real-variable description of the Thorin measure, and refinements of some known properties of the Lambert distribution.

1. Introduction

The principal Lambert function, denoted here by $W(s)$, is defined as the unique real-valued concave increasing solution to the functional equation

$$We^W = s. \tag{1.1}$$

This solution exists for $s \in [-e^{-1}, \infty)$ and it satisfies $W(-e^{-1}) = -1$, $W(0) = 0$, $W'(0) = 1$ and $W(s) \sim \log s$ as $s \to \infty$. The Lambert-W arises in various specific models by providing solutions to certain differential and functional equations. Corless et al. (1996) is the standard account of properties and applications of W. In addition, Brito et al. (2008), Caillol (2003) and Valluri et al. (2000), amongst others, describe a variety of applications. The standard reference Olver et al. (2010) classifies $W(s)$ as 'elementary' and it lists some of its properties.

The most interesting property of $W(s)$ for probability theory is that it is a Bernstein function, written $W \in \mathcal{B}$, and meaning that it has the integral representation

$$W(s) = \int_0^\infty \left(1 - e^{-sv}\right) \Omega(dv), \tag{1.2}$$

where Ω is a Lévy measure, i.e., $\Omega(\{0\}) = 0$ and $\int_0^\infty (v \wedge 1)\Omega(dv) < \infty$. It follows that there is a subordinator, i.e., a positive-valued Lévy process $(\Lambda_t : t \geq 0)$ which necessarily has increasing sample paths with $\Lambda_0 = 0$, and whose one-dimensional

E-mail address: tony.pakes@uwa.edu.au.

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laws satisfy

$$E\left(e^{-s\Lambda_t}\right) = e^{-tW(s)} = \left(\frac{W(s)}{s}\right)^t.$$

There are several published proofs of (1.2). Two of them depend ultimately on complex variable methods and they yield deeper results than the real-variable proofs. Kalugin et al. (2012) show that $W(s)/s$ has a Stieltjes transform representation, implying that $W(s)$ is a complete Bernstein function, written $W \in \mathcal{CB}$, and meaning that the Lévy measure Ω has a completely monotone density, denoted here by $\omega(y)$. Pakes (2011) independently shows that $W'(s)$ has a Stieltjes transform representation and hence that $W(s)$ is Thorin Bernstein, written $W \in \mathcal{TB}$, and meaning that $y\omega(y)$ is completely monotone. Clearly $\mathcal{TB} \subset \mathcal{CB}$ and the inclusion is strict. See Schilling et al. (2012) for much more about these Bernstein function classes.

The property $W \in \mathcal{TB}$ has the significant consequence that the probability laws $L(\Lambda_t)$ are self-decomposable (S.D.) for each $t > 0$, an aspect explored by Pakes (2011). We remark that this S.D. property in fact is an immediate consequence of the differentiation identity

$$W'(s) = \frac{W(s)}{s(1+W(s))} \tag{1.3}$$

and knowing that $W \in \mathcal{B}$. We show this in Section 4.

It is expedient at this point to summarise in the following proposition the results in Pakes (2011) which are relevant to this study.

Proposition 1.1. (i) *There is a probability measure* ν *satisfying* $\operatorname{supp}(\nu) = [e^{-1}, \infty)$ *and*

$$W'(s) = \int_{e^{-1}}^{\infty} \frac{\nu(dy)}{y+s}. \tag{1.4}$$

Hence ν *is the Thorin measure of* W*, i.e.,*

$$W(s) = \int_{e^{-1}}^{\infty} \log(1+s/y)\nu(dy) \tag{1.5}$$

yielding the Bernstein representation

$$W(s) = \int_0^{\infty} \left(1-e^{-sv}\right)\omega(v)dv, \tag{1.6}$$

where

$$\omega(v) = v^{-1}\int_{e^{-1}}^{\infty} e^{-vy}\nu(dy). \tag{1.7}$$

(ii) *Define* $\Lambda = \Lambda_1$ *and let* ε, U *and* Z *be independent random variables having, respectively, the standard exponential and uniform laws and, for* $z \geq 0$,

$$P(Z \leq z) = 1\left(z \geq e^{-1}\right)\int_0^z y^{-1}\nu(dy).$$

Then

$$\Lambda \stackrel{d}{=} \varepsilon U/Z. \tag{1.8}$$

In addition,

$$E(Z^{-n}) = \frac{(n+1)^n}{n!}, \qquad P(\Lambda > z) = o\left(z^{-\frac{1}{2}}e^{-z/e}\right), \quad (z \to \infty). \tag{1.9}$$

(iii) *The S.D. law* $L(\Lambda)$ *has the background driving Lévy process (BDLP) representation*

$$\Lambda = \int_0^{\infty} e^{-t}dC_t,$$

where $(C_t : t \geq 0)$ *is a compound Poisson process with unit jump rate and jump increments having the density function* $y\omega(y)$.

Finally,

$$W'(s) = E\left(\frac{Z}{Z+s}\right). \tag{1.10}$$

Part (i) comprises the essence of Theorem 3.1 in Pakes (2011), in particular, equations (3.7), (3.8), (3.1) and (3.10) there. Part (ii) is covered by Theorem (3.4) and its proof and Part (iii) is Theorem 3.3, both in Pakes (2011).

In this paper we refine many of the properties listed in Proposition 1.1 by using the integral evaluation

$$\int_0^{\pi} (\phi(x))^r dx = \frac{\pi r^r}{\Gamma(1+r)}, \qquad (r \geq 0) \tag{1.11}$$

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