Statistics and Probability Letters 139 (2018) 84–89

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

On the regularity of weak solutions to space–time fractional stochastic heat equations

Guang-an Zou [a], Guangying Lv [a], Jiang-Lun Wu [b,*]

[a] School of Mathematics and Statistics, Henan University, Kaifeng 475004, PR China
[b] Department of Mathematics, Swansea University, Swansea SA2 8PP, United Kingdom

ARTICLE INFO

Article history:
Received 1 November 2017
Received in revised form 30 March 2018
Accepted 4 April 2018
Available online 14 April 2018

Keywords:
Space–time fractional derivative
Stochastic heat equations
Weak solutions
Regularity properties

ABSTRACT

This study is concerned with the space–time fractional stochastic heat-type equations driven by multiplicative noise, which can be used to model the anomalous heat diffusion in porous media with random effects with thermal memory. We first deduce the weak solutions to the given problem by means of the Laplace transform and Mittag-Leffler function. Using the fractional calculus and stochastic analysis theory, we further prove the pathwise spatial–temporal regularity properties of weak solutions to this type of SPDEs in the framework of Bochner spaces.

1. Introduction

We consider the following space–time fractional stochastic partial differential equations (SPDEs) on a bounded domain $D \subset \mathbb{R}^d$ $(d \geq 1)$:

$$\begin{cases} \partial_t^\beta u(x,t) = -(-\triangle)^{\frac{\alpha}{2}} u(x,t) + I_t^{1-\beta}[\sigma(u(x,t))\dot{W}(x,t)], \ x \in D, \ t > 0, \\ u(x,t)|_{\partial D} = 0, t > 0, \\ u(x,0) = u_0(x), x \in D, \end{cases} \tag{1.1}$$

where ∂_t^β is the Caputo fractional derivative with $\beta \in (0,1)$, $(-\triangle)^{\frac{\alpha}{2}}$ is the fractional Laplacian with $\alpha \in (0,2]$, $I_t^{1-\beta}$ is the fractional integral operator will be given below. The dimension d and the parameters α and β in (1.1) satisfy that $d < \min\{2, \beta^{-1}\}\alpha$. Denote by $\dot{W}(x,t)$ space–time white noise modeling the random effects, and the function $\sigma : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz continuous function.

For any $\beta \geq 0$, we define the function $G_\beta(t) : \mathbb{R} \to \mathbb{R}$ by

$$G_\beta(t) = \begin{cases} \dfrac{1}{\Gamma(\beta)} t^{\beta-1}, & t > 0, \\ 0, & t \leq 0, \end{cases} \tag{1.2}$$

where $G_0(t) = 0$ and $\Gamma(\beta)$ denotes the gamma function. The Riemann–Liouville fractional integral operator I_t^β is defined by

$$I_t^\beta f(t) = (G_\beta * f)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)ds, \ t > 0, \tag{1.3}$$

* Corresponding author.
E-mail address: j.l.wu@swansea.ac.uk (J.-L. Wu).

https://doi.org/10.1016/j.spl.2018.04.006

with $I_t^0 f(t) = f(t)$. For $\beta \in (0, 1)$ and $t > 0$, then the expression

$$\partial_t^\beta f(t) = \frac{d}{dt}[I_t^{1-\beta}(f(t) - f(0))] = \frac{d}{dt}\left(\int_0^t G_{1-\beta}(t-s)(f(t) - f(0))ds\right) \tag{1.4}$$

is called the Caputo fractional derivative of order β of the function f (see Srivastava and Trujillo, 2006a).

Note that Eqs. (1.1) might be used to model the random effects on transport of particles in medium with thermal memory. Chen et al. (2015) introduced a class of SPDEs with time-fractional derivatives and proved the existence and uniqueness of solutions to the equations. Mijena and Nane (2015) proved the existence and uniqueness of mild solutions to non-linear space–time fractional SPDEs, and they also investigated the bounds for the intermittency fronts solutions of these equations (Mijena and Nane, 2016). Foondun and Nane (2015) studied the asymptotic properties of space–time fractional SPDEs. Chen et al. (2016) proved the existence and uniqueness of solutions to space–time fractional SPDEs in Gaussian noisy environment. It was worth mentioning that the above authors mainly focused on the mild solutions based on the Green function. As we known, the regularity of weak solutions to the fractional SPDEs has received less attention. The aim of this paper is to study the regularity of weak solutions to space–time fractional SPDEs, which are needed for the error analysis of numerical methods, for example, Zou et al. (2018) and Zou (2018) investigated the finite element methods for solving a special case of fractional SPDEs in the given problem (1.1).

The remaining of this paper is organized as follows. In Section 2, some notations and preliminaries will be introduced, and we also deduce the weak solutions to the space–time fractional SPDEs. In Section 3, stochastic analysis techniques and fractional calculus are used to prove the spatial and temporal regularity properties of weak solutions to Eq. (1.1) in Bochner spaces.

2. Notations and preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a filtered probability space with the normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Recall that Q is a positive bounded linear operator on some Hilbert space U with finite trace. Let $W = \{W(t), t \geq 0\}$ be a U-valued Wiener process with covariance operator Q. We introduce the subspace $U_0 = Q^{1/2}(U)$, which endowed with the inner product:

$$(u, v)_{U_0} = (Q^{1/2}u, Q^{1/2}v), \ u, v \in U_0,$$

and induced norm $\|\cdot\|_{U_0}$, where $Q^{-1/2}$ denotes the pseudo-inverse of $Q^{1/2}$. Denote by $L_2^0 = L^2(U_0, H)$ the space of Hilbert–Schmidt operators $T : U_0 \to H$ endowed with the norm

$$\|\varphi\|_{L_2^0}^2 = Tr[(\varphi Q^{1/2})(\varphi Q^{1/2})^*] < \infty,$$

for any $\varphi \in L_2^0$. The details description of Wiener process should be referred to Prévôt and Röckner (2007). Let $p \geq 2$ and $\{v(t)\}_{t\in[0,T]}$ be an L_2^0-valued predictable stochastic process, the following generalized version of Itô isometry (including the Burkholder–Davis–Gundy inequality) are important for the stochastic integrals (Kruse, 2014), that is

$$\mathbb{E}\|\int_0^t v(s)dW(s)\|^p \leq C(p)\mathbb{E}[(\int_0^t \|v(s)\|_{L_2^0}^2 ds)^{\frac{p}{2}}], \ t \in [0, T], \tag{2.1}$$

where $\mathbb{E}$ denotes the expectation and $C(p) > 0$ is a constant.

Next, we shall introduce fractional order spaces and norms. Define a linear operator $A := -\Delta$ with zero Dirichlet boundary condition on D. Denote by $\{\varphi_k\}_{k\geq 1}$ the complete orthonormal system of eigenfunctions in H for the operator A, i.e., for $k = 1, 2, \ldots,$

$$A\varphi_k = \lambda_k \varphi_k, \ \varphi_k|_{\partial D} = 0,$$

with $\lambda_1 \leq \lambda_2 \leq \cdots, \lambda_k \leq \cdots$.

For any $s > 0$, let $\dot{H}^s$ be the domain of the fractional power $A^{\frac{s}{2}} = (-\triangle)^{\frac{s}{2}}$, which can be defined by

$$A^{\frac{s}{2}}v = \sum_{k=1}^{\infty} \lambda_k^{\frac{s}{2}}(v, \varphi_k)\varphi_k, \ \dot{H}^s = \mathcal{D}(A^{\frac{s}{2}}) = \{v \in H : \|A^{\frac{s}{2}}v\|^2 = \sum_{k=1}^{\infty} \lambda_k^s (v, \varphi_k)^2 < \infty\},$$

with inner product $(u, v)_{\dot{H}^s} = (A^{\frac{s}{2}}u, A^{\frac{s}{2}}v)$ and induced norms $\|v\|_{\dot{H}^s}^2 = \|A^{\frac{s}{2}}v\|^2 = \sum_{k=1}^{\infty}\lambda_k^s(v, \varphi_k)^2$. It is known that $\dot{H}^0 = H$, $\dot{H}^1 = H_0^1(D)$ and $\dot{H}^2 = H^2(D) \cap H_0^1(D)$ with equivalent norms and that $\dot{H}^{-s}$ can be identified with the dual space $(\dot{H}^s)^*$ for $s > 0$. In order to quantify the regularity we introduce the Bochner spaces $L^p(\Omega; B) = L^p((\Omega, \mathcal{F}, \mathbb{P}); B)$ endowed with the norm:

$$\|v\|_{L^p(\Omega;B)} = (\mathbb{E}[\|v\|_B^p])^{\frac{1}{p}}, \forall \ v \in L^p(\Omega; B),$$

where B being a Banach space and for any $p \geq 2$.

For the sake of convenience, Eqs.(1.1) can be rewritten as the following abstract formulation:

$$\begin{cases} \partial_t^\beta u(t) + A_\alpha u(t) = I_t^{1-\beta}[\sigma(u(t))\dot{W}(t)], t > 0, \\ u(0) = u_0, \end{cases} \tag{2.2}$$