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Minimizing the probability of ruin: Two riskless assets with transaction costs and proportional reinsurance

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ABSTRACT

We compute the optimal investment and reinsurance strategy for an insurance company that wishes to minimize its probability of ruin, when the risk process follows Brownian motion with drift and when the insurer can buy proportional reinsurance. The financial market in which the insurer invests consists of two riskless assets. One riskless asset is a money market, and the insurer pays claims from the money market account. The other riskless asset is a bond that earns a higher interest rate than the money market, but buying and selling bonds are subject to proportional transaction costs.

1. Introduction

Many insurers employ an *integrated* investment and reinsurance strategy to increase profits and to protect themselves against reinsurable claims. Stochastic optimal control theory provides both theoretical and practical solutions to such optimal investment and reinsurance problems. Early work using this approach appears in Browne (1995), who considers a continuous-time diffusion model for the surplus of an insurer and derives an optimal investment strategy by maximizing the expected exponential utility of terminal wealth and also minimizing the probability of ruin. Since Browne's work in 1995, applying stochastic control to solve problems of interest in actuarial mathematics has become a small industry; see, for example, Hipp and Taksar (2010), Irgens and Paulsen (2004), Li et al. (2017), Luo and Taksar (2011), Promislow and Young (2005), Schmidli (2002), Zhang and Siu (2009) and Zhang and Liang (2016), to name just a few.

Most of above-referenced research allows insurers to invest in financial markets *without* transaction costs because models with transaction costs are difficult to handle. Research on optimal control in the presence of transaction costs includes Azcue and Muler, 2013; Davis and Norman, 1990; Shreve and Mete Soner, 1994; Shreve et al., 1991; Weerasinghe Ananda, 1998. Davis and Norman (1990) and Shreve and Mete Soner (1994) study an optimal investment and consumption model for a power-utility-maximization problem with proportional transaction costs in a Black–Scholes market. An alternative financial model to one riskless and one or more risky assets is a financial market with *two riskless* assets. The two riskless assets earn rates r and R with $0 < r < R$, and trading the asset earning rate R is subject to proportional transaction costs. Shreve et al. (1991) apply this model to maximize expected utility of lifetime consumption, and Azcue and Muler (2013) apply this model to minimize the probability of insurer ruin.

In this paper, we find the optimal investment and reinsurance strategy for an insurer who invests its surplus in a financial market consisting of two riskless assets, which seems to us to be particularly relevant for insurers whose assumption of

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financial risk can be limited due to regulation.[1] One riskless asset is a money market, and the insurer pays claims from that account. The other riskless asset is a bond that earns a higher interest rate than the money market, but which is subject to proportional transaction costs whenever the insurer buys or sells bonds. In addition to investing in these two riskless assets, the insurer can control its insurance risk by purchasing reinsurance. The objective of the insurer is to choose an optimal investment and reinsurance strategy to minimize its probability of ruin.

The remainder of the paper is organized as follows. In Section 2.1, we describe the financial market in which the insurer invests and purchases reinsurance, we define the problem of minimizing the probability of ruin, and we prove some basic results about that probability of ruin. In Section 2.2, we consider the case for which the rate of return on the bond is large enough that short-selling the bond is never optimal. We present a verification theorem that we use to obtain the minimum probability of ruin. In Section 2.3, we consider the case for which the rate of return on the bond is small enough that short-selling the bond might be optimal. Again, we use a verification theorem to obtain the minimum probability of ruin. Section 3 concludes this paper.

2. Ruin with no limit on borrowing from the money market or on shorting the bond

2.1. Model formulation and preliminary results

As in Promislow and Young (2005), we model the claim process C according to Brownian motion with drift. Specifically,

$$dC_t = a\,dt - b\,dB_t,$$

in which a and b are positive constants and B is a standard Brownian motion on the filtered probability space $\big(\Omega, \mathcal{F}, \{F_t\}_{t\geq 0}, \mathbb{P}\big)$. The Brownian motion model of the claim process is a limit of the classical compound Poisson model; see, for example, Grandell (1991). We assume that premium is payable continuously at the rate $c = (1+\theta)a$, for some $\theta \geq 0$, that is, the insurer charges premium according to the expected value principle. Thus, before introducing investments and reinsurance, the surplus process U follows the dynamics

$$dU_t = c\,dt - dC_t = \theta\,a\,dt + b\,dB_t.$$

We assume that there are two riskless assets available to the insurer in the financial market, the interest rates of which are constants r and R, with $0 < r < R$. We refer to the high-yield riskless asset as the "bond", and the low-yield riskless asset as the "money market (account)". The insurer can transfer funds between the bond and the money market subject to proportional transaction costs payable from the money market. Specifically, to invest \$1 in the bond, the insurer must pay $\$(1+\lambda)$ from the money market, with $\lambda > 0$; similarly, after selling \$1 of the bond, the insurer will only receive $\$(1-\mu)$ to invest in the money market, with $0 < \mu < 1$.

Let M_t and N_t represent the cumulative purchase and sale, respectively, of the bond, at time $t \geq 0$. A buying strategy $\{M_t\}$ (respectively, selling strategy $\{N_t\}$) is *admissible* if it is non-negative, non-decreasing, right continuous with left limits (RCLL), and adapted to the filtration $\{F_t\}$. Let X_t and Y_t denote the amount invested in the money market and bond, respectively, at time t.

In addition to controlling surplus through investing in two riskless asset, we also assume the insurer can buy proportional reinsurance, with premium computed according to the expected value principle with proportional risk loading $\eta > \theta$. Let q_t denote the proportion of claims *retained* by the insurer. A retention strategy $\{q_t\}$ is *admissible* if it is non-negative, RCLL, and adapted to the filtration $\{F_t\}$, and if $\int_0^t q_s^2 ds < \infty$ with probability 1 for all $t \geq 0$.[2] The set of admissible triples of strategies $\{M, N, q\}$ is denoted by $\mathcal{A}$.

Then, given the initial portfolio $(X_{0-}, Y_{0-}) = (x, y)$ and a triple of controls $\{M, N, q\} \in \mathcal{A}$, the insurer's surplus follows the dynamics

$$\begin{cases} dX_t = \Big(rX_t + R\,Y_t + \big(\theta - \eta(1-q_t)\big)a\Big)dt + b\,q_t\,dB_t - \big(1+\lambda\big)dM_t + \big(1-\mu\big)dN_t, \\ dY_t = dM_t - dN_t. \end{cases} \tag{2.1}$$

Note that we assume the bond is interest bearing with the interest payable into the money market account. By contrast, in Shreve et al. (1991) and Liang and Young (2017), the bond is treated as a zero-coupon bond with the value of the bond itself increasing at rate R; thus, in that work, $dY_t = R\,Y_t\,dt + dM_t - dN_t$.

Let $L = L(x, y)$ denote the liquid value of the portfolio (x, y); specifically,

$$L(x, y) = x + (1-\mu)y^+ - (1+\lambda)y^-.$$

Let $\{X_t, Y_t\}$ denote the solution of (2.1) under an admissible strategy, with $X_{0-} = x$ and $Y_{0-} = y$, and define the corresponding time of ruin by[3]

$$\tau_0 = \inf\Big\{t \geq 0 \,\Big|\, L\big(X_t, Y_t\big) < 0\Big\}. \tag{2.2}$$

[1] Our model is closely related to the one in Azcue and Muler (2013), but they do not consider reinsurance.

[2] Note that we allow $q_t > 1$. If $q_t > 1$, then the insurer is increasing its risk/business, and one can interpret η as the proportional benefit of adding business; see, for example, Bäuerle (2005).

[3] We simplify notation by omitting the dependence of X_t, Y_t, and τ_0 on the admissible strategy.