Statistics and Probability Letters xx (xxxx) xxx–xxx

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Glivenko–Cantelli Theorem for the kernel error distribution estimator in the first-order autoregressive model

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ARTICLE INFO

Article history:
Received 26 January 2018
Accepted 26 March 2018
Available online xxxx

MSC:
62G20
60F15

Keywords:
Kernel estimator
Glivenko–Cantelli Theorem
CDF. Residuals
Autoregressive models

ABSTRACT

This paper considers the uniform strong consistency of the kernel estimator of the error cumulative distribution function (CDF) in the first-order autoregressive model. The classical Glivenko–Cantelli Theorem is extended to the residual based kernel smooth CDF estimator in the autoregressive model.

1. Introduction

Let the random variables $X_0, X_1, X_2, \ldots, X_n$ be generated by the first-order autoregressive (AR(1)) model

$$X_i = \rho X_{i-1} + \varepsilon_i\,, \tag{1.1}$$

where ε_i are independent and identically distributed (i.i.d.) random variables with mean 0, finite second moment and unknown cumulative distribution function (CDF) F. We also assume that $|\rho| < 1$ and the sequence $\{X_i\}$ is stationary. Then we have the representation

$$X_i = \sum_{j=0}^{\infty} \rho^j \varepsilon_{i-j}.$$

and

$$E(|X_i|) < \infty. \tag{1.2}$$

About the estimation of the error distribution, Lee and Na (2002) and Bachmann and Dette (2005) extend the asymptotic result (based on the L_2-norm) in Bickel and Rosenblatt (1973) to the error density estimator in AR(1) model (1.1) while Horváth and Zitikis (2003) consider asymptotics of the L_p-norms of the estimators. Cheng (2005) shows the asymptotic distribution of the kernel error density estimator at a fixed point and globally.

In this paper, we further consider the uniform strong consistency of the CDF estimator in model (1.1). We extend the classical Glivenko–Cantelli Theorem to the residual based kernel smooth distribution function. For notation simplicity

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https://doi.org/10.1016/j.spl.2018.03.018

we only consider the first-order autoregressive AR(1) model (1.1), the current results can be extended to the ordinary autoregressive AR(p) model, $p > 1$, as long as similar conditions are imposed.

The outline of this paper is as follows. Section 2 introduces estimators, some basic assumptions and the main results. Section 3 provides the details of the proofs.

In the following sections, all limits are taken as the sample size n tends to ∞.

2. Estimators, assumptions and the main results

In this section we first introduce the estimator of ρ and the kernel estimator of the CDF F in model (1.1).

Assume that $|\rho| < 1$ and $0 < E(\epsilon^2) < \infty$, and we observe $X_0, X_1, X_2, \ldots, X_n$. Let $\hat{\rho}$ be an estimator of ρ with the following almost surely (a.s.) property:

$$|\hat{\rho} - \rho| \leq Cn^{-1/2}(\ln_2 n)^{1/2}, \ \text{a.s.}, \tag{2.1}$$

where $\ln_2 n = \ln(\ln n)$ and $C < \infty$ is a constant.

Remark. The well-known least square estimator satisfies the strong consistency property (2.1), which can be verified by Theorem 2 in Koul and Zhu (1995).

Note that $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are i.i.d. with the unknown CDF $F(t)$. Let F_n denote the empirical distribution function, i.e.,

$$F_n(t) = \frac{1}{n}\sum_{i=1}^{n} I(\varepsilon_i \leq t), \ t \in R^1,$$

where I denotes the indicator function.

The classical Glivenko–Cantelli theorem says that $F_n(t)$ converges almost surely (a.s.) to $F(t)$ uniformly in $t \in R^1$, i.e.,

$$\sup_{t\in R^1}|F_n(t) - F(t)| \to 0, \ \text{a.s.} \tag{2.2}$$

Notice that F_n is infeasible for model (1.1), since ε_i, $1 \leq i \leq n$ are not observable. Let

$$\hat{\varepsilon}_i = X_i - \hat{\rho}X_{i-1}, \quad 1 \leq i \leq n \tag{2.3}$$

denote the residuals. Based on these residuals, we construct empirical distribution function $\hat{F}_n$ as follows:

$$\hat{F}_n(t) = \frac{1}{n}\sum_{i=1}^{n} I(\hat{\varepsilon}_i \leq t), \ t \in R^1.$$

But there is one drawback of $\hat{F}_n$ which is its discontinuity, regardless of F being continuous or discrete. To remedy this deficiency of $\hat{F}_n$, similar to the kernel smooth distribution estimator proposed in Yamato (1973), here we define the residual based kernel distribution estimator $\tilde{F}$ by

$$\tilde{F}(t) = \int_{-\infty}^{t} \frac{1}{n}\sum_{i=1}^{n} k_h\left(u - \hat{\varepsilon}_i\right) du = \frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{t} k_h\left(u - \hat{\varepsilon}_i\right) du, \ t \in R^1 \tag{2.4}$$

in which $h = h_n > 0$ is called bandwidth, k is an integrable function called kernel, and $k_h(u) = k(u/h)/h$.

To derive the uniform strong convergence of $\tilde{F}$ for F, we need the following assumptions on the kernel k.

Assumption k. Functions $k(x)$, $xk(x)$ are integrable on the whole real line, and

$$\int_{-\infty}^{+\infty} k(x)dx = 1. \tag{2.5}$$

The main results are the uniform closeness between $\hat{F}_n$ and F_n, and the Glivenko–Cantelli Theorem for the estimator $\tilde{F}_n$.

Theorem 2.1. *Assume that ε_i has finite second moment, and (2.1) holds. Then, for the continuous CDF F with bounded first order derivative, i.e., there exists a constant c_1 $(0 \leq c_1 < \infty)$ such that*

$$|F'(t)| \leq c_1 \ \text{for any} \ t \in R^1, \tag{2.6}$$

we have that

$$\sup_{t\in R^1}\left|\hat{F}_n(t) - F_n(t)\right| \to 0. \ \ a.s. \tag{2.7}$$

Please cite this article in press as: Cheng F., Glivenko–Cantelli Theorem for the kernel error distribution estimator in the first-order autoregressive model. Statistics and Probability Letters (2018), https://doi.org/10.1016/j.spl.2018.03.018.