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On optimal stopping and free boundary problems under ambiguity

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ABSTRACT

This paper considers verification theorems for optimal stopping under ambiguity for diffusion processes via connecting with free boundary problems in infinite time horizon.

1. Introduction

Every decision is a risky business. Finding an optimal time to execute is of importance in many situations. The classical specification of optimal stopping is to find a stopping time τ_* for a payoff process $\{\xi_t\}_{t\geq 0}$ such that $\mathbb{E}_P[\xi_{\tau_*}] = \sup_\tau \mathbb{E}_P[\xi_\tau]$, see Øksendal (2006), where the supremum is taken over all stopping times τ for $\{\xi_t\}_{t\geq 0}$, and $\mathbb{E}_P$ denotes the expectation taken with respect to the probability law P. It is known that the solution of a free boundary problem yields the solution of an optimal stopping problem for diffusion processes, see Mckean (1965) and Peskir and Shiryaev (2006).

The knowledge about the likelihood of future states is often assumed to be characterized by a plausible probability measure in financial markets (see Peskir and Shiryaev (2006)), because the transaction motivation is considered to be caused by different risk preferences rather than belief difference. In real decision making scenarios, however, the influence of heterogeneous beliefs cannot be ignored. Economic agents absolutely do not agree that a single probability can capture future market's ambiguity. Indeed, the ambiguity is not described by a single probability measure, but by a set of probability measures. Several models are developed in Gilboa and Schmeidler (1989), Chen and Epstein (2002), Nishimura and Ozaki (2007) and Zhao (2009). It has been proved that the g-expectation induced by a backward stochastic differential equation (BSDE for short) by Peng (1997) can describe the degree of ambiguity tolerance, such as the worst case and the case of ambiguity aversion with a penalty term, see Chen and Epstein (2002), Delbaen et al. (2010) and Chen et al. (2013). An optimal stopping rule under ambiguity in discrete time was developed by Riedel (2009). As a continuous counterpart, Chen et al. (2013) considered the optimal stopping problem in continuous time.

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In this article, by applying g-expectation as a technical setting to represent the ambiguity, we extend the verification theorem to ambiguity framework in infinite time horizon, and propose two kinds of verification theorems with different aggregators g and boundary conditions.

2. Main results

Let $\{W_t\}_{t\geq 0}$ be a standard d-dimension Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$, and the filtration $\mathcal{F}_t = \sigma(W_s, s \leq t)$. P is a reference probability measure.

Given $g(y, z, x) : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, we make the following assumptions:

(H1) Lipschitz condition: there exists a constant $L > 0$ such that

$$|g(y_1, z_1, x) - g(y_2, z_2, x)| \leq L(|y_1 - y_2| + \|z_1 - z_2\|), \text{ for } (y_i, z_i) \in \mathbb{R} \times \mathbb{R}^d, \ i = 1, 2;$$

The norm $\|\cdot\|$ is given by $\|z\| = \sqrt{z^* z}$, where z^* is the transpose of vector z.

(H1′) $g(y, z, x)$ is continuously differentiable with respect to (y, z, x), and the derivatives are bounded;

Obviously, (H1′) implies (H1).

(H2) Monotonicity condition: for some $a \in \mathbb{R}$, we have

$$(y_1 - y_2)[g(y_1, z, x) - g(y_2, z, x)] \leq -a|y_1 - y_2|^2, \text{ for } (y_i, z) \in \mathbb{R} \times \mathbb{R}^d, \ i = 1, 2;$$

(H3) Random terminal constraint: for a finite stopping time τ, given $\xi_\tau \in \mathcal{F}_\tau$,

$$\mathbb{E}_P[e^{\rho\tau}(|\xi_\tau|^2 + 1)] < \infty, \text{ for some } \rho > L^2 - 2a;$$

(H4) $g(y, 0, x) \equiv 0$, for any $(y, x) \in \mathbb{R} \times \mathbb{R}^d$.

Assume $\{X_t\}_{t\geq 0}$ describes the dynamics of an underlying asset,

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \ X_0 = x, \ x \in \mathbb{R}^d, \tag{1}$$

where the $\mathbb{R}^d$-valued (resp. $\mathbb{R}^{d\times d}$-valued) function b (resp. σ) satisfies $b \in C_b^1$ (resp. $\sigma \in C_b^2$), and C^i (resp. C_b^i) denotes the set of functions with continuous (resp. bounded) derivatives up to order i.

(H5) There exist two constants $c_2 \geq c_1 > 0$ such that

$$c_1 \sum_{i=1}^d \eta_i^2 \leq \frac{1}{2} \sum_i^d \sum_j^d (\sigma\sigma^*)_{ij}(x)\eta_i\eta_j \leq c_2 \sum_{i=1}^d \eta_i^2, \ \eta = (\eta_1, \ldots, \eta_d) \in \mathbb{R}^d.$$

(H5′)

$$\frac{1}{2} \sum_i^d \sum_j^d (\sigma\sigma^*)_{ij}(x)\eta_i\eta_j > c_1 \sum_{i=1}^d \eta_i^2, \ \eta = (\eta_1, \ldots, \eta_d) \in \mathbb{R}^d.$$

Let

$$\mathcal{L} := \sum_i^d b_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_i^d \sum_j^d (\sigma\sigma^*)_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j},$$

where σ^* is the transpose of matrix σ.

If (H1)–(H4) hold, Darling and Pardoux (1997) deduced the following BSDE with random terminal time τ

$$y_t = \xi_\tau + \int_{t\wedge\tau}^{\tau} g(y_s, z_s, X_s)ds - \int_{t\wedge\tau}^{\tau} z_s dW_s \tag{2}$$

has a unique solution $(y_t, z_t)_{t\in[0,\tau]}$.

Now, we introduce the notion of g-expectation with random terminal time. It is a natural extension of the classical expectation, and preserves most of properties of the classical expectation except linearity, with reference to Peng (1997).

Definition 1. The g-expectation $\mathcal{E}_g[\cdot]$ is defined by $\mathcal{E}_g[\xi_\tau] = y_0$, where y_0 can be calculated by setting $t = 0$ in BSDE (2).

Given a bounded domain G in $\mathbb{R}^d$, we use $\overline{G}$ to denote its closed hull. Let $\tau_G := \inf\{t > 0, X_t \notin G\}$ be the first exit time from the set G of X.

A class of optimal stopping problems under ambiguity involving a continuous payoff function $\Phi = \Phi(x) : G \to [0, \infty)$ is the problem of searching for a stopping time $\tau_* \leq \tau_G$ such that

$$\mathcal{E}_g[\Phi(X_{\tau_*})] = \sup_{\tau\leq\tau_G} \mathcal{E}_g[\Phi(X_\tau)]. \tag{3}$$

Inspired by the nonlinear Feynman–Kac formula for fixed boundary value problems in Peng (1991), we give a verification theorem for optimal stopping via connecting with free boundary problems.