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On the non-monotonic analogue of a class based on the hazard rate order

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ABSTRACT

Weak convergence and related issues as well as reliability and moment bounds are studied for a non-monotonic analogue of the NBUFR class of life distributions. Connections of this new family with existing non-monotonic ageing classes are explored. Certain properties of the NBUFR family, hitherto unknown, are obtained as special cases.

1. Introduction

Consider a non-negative random variable X with *cumulative distribution function* (cdf) $F(t)$ and *survival function* $\bar{F}(t) := 1 - F(t)$. For absolutely continuous X with *probability density function* (pdf) $f(t)$, the *hazard rate (failure rate) function* is defined as $r_F(t) := f(t)/\bar{F}(t)$ for all t for which $\bar{F}(t) > 0$.

The hazard rate function has widespread applications—it is used by actuaries under the name of 'force of mortality' to compute mortality tables; in statistics, its reciprocal for the Gaussian distribution is popularly known as 'Mills' ratio' and its use can be traced back to Laplace. It plays an important role in determining the form of extreme value distributions and in extreme value theory it is known as 'intensity function'. In view of its importance in reliability and survival analysis, Abouammoh and Ahmed (1988) introduced a class based on the comparison of failure rates of a new and a used item leading to a generalization of the NBU class. One has the following formal definition:

Definition 1.1. An absolutely continuous probability distribution function F on $[0, \infty)$ for which $F(x)/x$ has a limit as $x \to 0+$ is called *New Better than Used in Failure Rate* (NBUFR) if $\bar{r}_F(0) \le r_F(t)\ \ \forall\ t > 0$ where $\bar{r}_F(0) := \lim_{t\to 0+} t^{-1} \int_0^t r_F(x)dx$.

Its dual, the *New Worse than Used in Failure Rate* (NWUFR) class, is defined by reversing the direction of the above inequality. It is implicit in the above definition that $\bar{r}_F(0) > 0$; otherwise, the classes fail to have any sensible ageing interpretations. *Here and in all that follows, we will, without loss of generality, consider the pdf to be right continuous at zero.* If not, we shall define $f(0)$ to be $\bar{r}_F(0)$ (thereby ensuring right continuity at zero) and work with this version of the pdf. Note

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that altering the value of the pdf at a single point (a set of Lebesgue measure zero) does not affect any of the probabilistic properties of the distribution since it is assumed to be absolutely continuous with respect to the Lebesgue measure.

An alternative interpretation for the NBUFR class can be given from the standpoint of stochastic orderings, as follows: For two non-negative random variables X and Y with hazard rate functions $r_F(t)$ and $r_G(t)$ respectively, we say that X is said to be smaller than Y in the hazard rate order (denoted as $X \leq_{hr} Y$) iff $r_F(t) \geq r_G(t)\ \forall\ t \in \mathbb{R}$ (see, Shaked and Shanthikumar (1994) for details). This implies that a non-negative random variable X is said to be NBUFR if $X \leq_{hr} Y$, where $Y \sim Exp\,(\bar{r}_F(0))$ i.e. an exponential distribution with mean $1/\bar{r}_F(0)$.

The NBUFR family occupies a significant position among monotonic ageing classes as is evident from the following hierarchical relationships: $\{IFR\} \subset \{IFRA\} \subset \{NBU\} \subset \{NBUFR\}$. From a physical standpoint, the property is motivated by the fact that there is empirical evidence of the initial failure rate for a new component being nonzero (see Davis (1952)). Shock model theory and preservation properties, closure under coherent systems and testing problems have already been explored for the NBUFR class (see Abouammoh and Ahmed (1988), Gohout and Kuhnert (1997) and others).

As monotonic ageing notions are not sufficient to describe many real life scenarios, attempts have been made to model such situations by means of bathtub failure rates (see Glaser (1980), Lai and Xie (2006)) as well as IDMRL and NWBUE distributions (see Guess et al. (1986) and Mitra and Basu (1994)). In a recent paper, Pandey and Mitra (2011) considered a new notion of non-monotonic ageing by introducing the *New Worse then Better than Used in Failure Rate* (NWBUFR) class while studying Poisson shock models.

Definition 1.2. An absolutely continuous life distribution function F having support on $[0, \infty)$, for which $F(x)/x$ has a limit as $x \to 0+$, is called *New Worse then Better than Used in Failure Rate* (NWBUFR) if there exists a point $t_0 \geq 0$ such that

$$\bar{r}_F(0) \geq r_F(t), \qquad t < t_0 \text{ and } \bar{r}_F(0) \leq r_F(t), \qquad t \geq t_0 \tag{1.1}$$

where $\bar{r}_F(0)$ is defined as in Definition 1.1.

The point t_0, not necessarily unique, is called the *change point* or *turning point* of F in the NWBUFR sense; we write NWBUFR(t_0) to indicate this. The following is an example of a parametric family of NWBUFR distributions.

Example 1.1. Consider the life distribution having survival function:

$$\bar{F}(t) = \begin{cases} (t+1)^{-a}, & 0 \leq t < b, \\ \dfrac{(2b+1-t)^a}{(b+1)^{2a}}, & b \leq t < c, \\ \left(\dfrac{2b+1-c}{b+1}\right)^{2a} (2(b-c)+1+t)^{-a}, & c \leq t < d, \\ \dfrac{1}{(2(b-c)+1+d)^a}\left(\dfrac{2b+1-c}{b+1}\right)^{2a} \exp\Big(\dfrac{a(t-d)}{2(b-c)+1+d}\Big), & t \geq d, \end{cases}$$

where the parameters a, b, c and d are such that $a, b > 0$, $c > 2b$ and $2c > 2b + d$. The corresponding hazard rate function then works out as $r_F(t) = \frac{a}{t+1}\mathbf{I}_{[0,b)}(t) + \frac{a}{2b+1-t}\mathbf{I}_{[b,c)}(t) + \frac{a}{2(b-c)+1+t}\mathbf{I}_{[c,d)}(t) + \frac{a}{2(b-c)+1+d}\mathbf{I}_{[d,\infty)}(t)$.

Here, $\mathbf{I}_A(\omega)$ denotes the indicator function of the set A defined by $\mathbf{I}_A(\omega) = 1$, if $\omega \in A$ and $= 0$ if $\omega \notin A$. It is easy to observe that the life distribution F is NWBUFR with $t_0 = 2b$.

Interrelationships between various ageing classes and their probabilistic properties have been extensively explored in the literature (see Barlow and Proschan (1981) and Lai and Xie (2006) for details). In Section 2, we investigate the interrelationships of the NWBUFR family with other existing non-monotonic ageing classes, namely BFR, IDMRL and NWBUE. Reliability and moment bounds as well as a characterization theorem for the exponential distribution can be found in Section 3. In Section 4 we establish the closure of the NWBUFR family under weak convergence and also its equivalence to the convergence of moments of all orders to the corresponding moments of the limiting distribution. In fact, corresponding theorems hitherto unknown, for the NBUFR and NWUFR families are obtained as by-products of our results, thereby making new contributions to the theory of the monotonic ageing classes.

2. Relationships amongst NWBUFR, BFR, IDMRL and NWBUE classes

It is known that the BFR and IDMRL classes of life distributions are contained in the NWBUE family. Here we continue further investigation.

Proposition 2.1. *If a life d.f. F is BFR with change point t^*, then F is either NWUFR or NWBUFR(t_0) with $t_0 \geq t^*$.*

Proof. Since $\bar{r}_F(0) = r_F(0)$, from the BFR property of F, we have $r_F(t)$ is nonincreasing in $[0, t^*)$, i.e. $r_F(t) \leq \bar{r}_F(0)$ for $t < t^*$. Now for $t \geq t^*$, $r_F(t)$ is nondecreasing. Here two cases may arise—either $\exists$ a point $t_0 < \infty$ for which $r_F(t)$ exceeds $\bar{r}_F(0)$ and subsequently increases, implying that F is NWBUFR(t_0) with $t_0 \geq t^*$; alternatively, there will not exist any such finite t_0, i.e. $r_F(t)$ is increases but remains bounded above by $\bar{r}_F(0)\ \forall\ t \geq t_0$ so that F is NWUFR. $\square$

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