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Brillinger-mixing point processes need not to be ergodic

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A B S T R A C T

Recently, it has been proved that a stationary Brillinger-mixing point process is mixing (of any order) if its moment measures determine the distribution uniquely. In this paper we construct a family of non-ergodic stationary point processes as mixture of two distinct Brillinger-mixing Neyman–Scott processes having the same moment measures.

1. Introduction and preliminaries

A (random) *point process* (short: PP) Ψ on the Euclidean space $\mathbb{R}^d$ is defined to be a $(\mathcal{F},\mathcal{N})$-measurable mapping from a hypothetical probability space $[\Omega,\mathcal{F},\mathbf{P}]$ into the measurable space $[\mathsf{N},\mathcal{N}]$ of all locally finite counting measures ψ (with countable support $s(\psi)$) acting on the σ-algebra $\mathcal{B}^d$ of Borel sets in $\mathbb{R}^d$ equipped with the smallest σ-algebra $\mathcal{N}$ containing all the sets $\{\psi \in \mathsf{N} : \psi(B) = j\}$ for $j \in \mathbb{N} := \{0, 1, \ldots\}$ and any bounded $B \in \mathcal{B}^d$. We briefly write $\Psi \sim P$, where the probability measure $P = \mathbf{P} \circ \Psi^{-1}$ induced on $[\mathsf{N},\mathcal{N}]$ by the mapping Ψ is called the *distribution* of Ψ. A PP $\Psi \sim P$ is called *stationary* or *homogeneous* if $P(T_x Y) = P(Y)$ for all $x \in \mathbb{R}^d$ and $Y \in \mathcal{N}$, where $T_x Y := \{T_x\psi : \psi \in Y\}$ with $(T_x\psi)(\cdot) = \psi((\cdot) + x)$. Note that it suffices to check stationarity for all $Y = \{\psi \in \mathsf{N} : \psi(B_1) = k_1, \ldots, \psi(B_\ell) = k_\ell\}$ with bounded $B_1, \ldots, B_\ell \in \mathcal{B}^d$, $k_1, \ldots, k_\ell \in \mathbb{N}$, $\ell \geq 1$. We will briefly write $P \in \mathcal{P}_\lambda^\infty$ if $\Psi \sim P$ is stationary with positive *intensity* $\lambda = \mathbf{E}\Psi([0,1]^d)$ and $\mathbf{E}\Psi^k([0,1]^d) < \infty$ for all $k \geq 1$, where $\mathbf{E}$ denotes the expectation w.r.t. $\mathbf{P}$. Further we need the *probability generating functional* (short: PGF) of $\Psi \sim P$ defined for all Borel measurable $v \,|\, \mathbb{R}^d \mapsto [0,1]$ with $1 - v$ having bounded support by

$$G_P[v] := \int_{\mathsf{N}} \prod_{x \in s(\psi)} v(x)^{\psi(\{x\})}\, P(\mathrm{d}\psi)\,. \tag{1.1}$$

Note that already the family $G_P[1 + \sum_{i=1}^k (z_i - 1)\,\mathbf{1}_{B_i}]$ for $z_1, \ldots, z_k \in [0,1]$ and bounded, pairwise disjoint $B_1, \ldots, B_k \in \mathcal{B}^d$ and $k \geq 1$ determines the distribution P on $[\mathsf{N},\mathcal{N}]$. For a rigorous introduction into and further background of PP theory the reader is referred to Daley and Vere-Jones (2008).

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The Brillinger-mixing property of a PP $\Psi \sim P \in \mathcal{P}_\lambda^\infty$ is based on its higher-order factorial cumulant measures $\gamma_P^{(k)}$ for $k \geq 2$ which can be defined by means of (1.1) as follows:

$$\gamma_P^{(k)}\big(\mathop{\times}_{i=1}^{k} B_i\big) := \lim_{z_1,\ldots,z_k \uparrow 1} \frac{\partial^k}{\partial z_1 \cdots \partial z_k} \log G_P\big[1 + \sum_{i=1}^{k}(z_i - 1)\,\mathbf{1}_{B_i}\big] \tag{1.2}$$

for all bounded $B_1, \ldots, B_k \in \mathcal{B}^d$. The rule of logarithmic differentiation applied on the right-hand side of (1.2) yields the representation of $\gamma_P^{(k)}$ in terms of the factorial moment measures $\alpha_P^{(\ell)}$ of order $\ell = 1, \ldots, k$:

$$\gamma_P^{(k)}\big(\mathop{\times}_{i=1}^{k} B_i\big) = \sum_{\ell=1}^{k}(-1)^{\ell-1}\,(\ell-1)! \sum_{K_1\cup\cdots\cup K_\ell=\{1,\ldots,k\}} \prod_{j=1}^{\ell} \alpha^{(\#K_j)}\big(\mathop{\times}_{i\in K_j} B_i\big)\,, \tag{1.3}$$

for all bounded $B_1, \ldots, B_k \in \mathcal{B}^d$, where the inner sum $\sum$ stretches over all decompositions of the set $\{1, \ldots, k\}$ into disjoint subsets $K_1, \ldots, K_\ell$ with cardinality $\#K_j \geq 1$ for $j = 1, \ldots, \ell$, see also Leonov and Shiryaev (1959) and Krickeberg (1982). Here, the kth-order factorial moment measure $\alpha_P^{(k)}$ is defined by

$$\alpha_P^{(k)}\big(\mathop{\times}_{i=1}^{k} B_i\big) = \int_{\mathsf{N}}\int_{B_1}\int_{B_2}\cdots\int_{B_k}\big(\psi - \sum_{i=1}^{k-1}\delta_{x_i}\big)(\mathrm{d}x_k)\cdots(\psi - \delta_{x_1})(\mathrm{d}x_2)\,\psi(\mathrm{d}x_1)\,P(\mathrm{d}\psi) \tag{1.4}$$

with Dirac measure $\delta_x(\cdot)$ defined by $\delta_x(B) = \mathbf{1}_B(x)$ for $x \in \mathbb{R}^d$ and $B \in \mathcal{B}^d$. Note that (1.4) can be obtained from (1.1) if on the right-hand side of (1.2) the 'log' is omitted. By standard techniques from measure theory it is easily seen from (1.3) that $\gamma_P^{(k)}$ can be extended to a locally finite (in general) signed measure on $[(\mathbb{R}^d)^k, \mathcal{B}^{dk}]$.

Due to the stationarity of $\Psi \sim P \in \mathcal{P}_\lambda^\infty$ it follows from (1.4) and (1.3) that $\gamma_P^{(k)}(\times_{i=1}^{k}(B_i + x))$ does not depend on $x \in \mathbb{R}^d$. This means that, for any $k \geq 2$, there exists of a unique *reduced factorial cumulant measure* $\gamma_{P,red}^{(k)}$ on $[(\mathbb{R}^d)^{k-1}, \mathcal{B}^{d(k-1)}]$ satisfying the disintegration formula

$$\gamma_P^{(k)}\big(\mathop{\times}_{i=1}^{k} B_i\big) = \lambda \int_{B_k} \gamma_{P,red}^{(k)}\big(\mathop{\times}_{i=1}^{k-1}(B_i - x)\big)\,\mathrm{d}x\,. \tag{1.5}$$

According to the Hahn–Jordan decomposition of signed measures, for each $k \geq 2$, $\gamma_{P,red}^{(k)}$ can be written as difference $\gamma_{P,red}^{(k)} = \gamma_{P,red}^{(k)+} - \gamma_{P,red}^{(k)-}$ of two (positive) measures $\gamma_{P,red}^{(k)+}$ and $\gamma_{P,red}^{(k)-}$ both concentrated on two disjoint sets. The corresponding *total variation measure* $|\gamma_{P,red}^{(k)}|$ of $\gamma_{P,red}^{(k)}$ is then defined by the sum

$$|\gamma_{P,red}^{(k)}|(B) := \gamma_{P,red}^{(k)+}(B) + \gamma_{P,red}^{(k)-}(B) \quad \text{for} \quad B \in \mathcal{B}^{d(k-1)}\,.$$

The following weak-dependence condition for PPes has been introduced by D.R. Brillinger in Brillinger (1975) (see Brillinger, 1991 for a historical review) and used by many authors to derive asymptotic normality of shot-noise processes, see e.g. Heinrich and Schmidt (1985), and of various empirical functionals related with (factorial) moment measures defined on large sampling windows, see e.g. Biscio and Lavancier (2016), Guan and Sherman (2007), Heinrich and Klein (2014), Karr (1987) and Krickeberg (1982).

Cond(B): A PP $\Psi \sim P \in \mathcal{P}_\lambda^\infty$ is said to be *Brillinger-mixing* if the signed measures $\gamma_{P,red}^{(k)}$ have finite total variation $|\gamma_{P,red}^{(k)}|(\mathbb{R}^{d(k-1)})$ for all $k \geq 2$.

Besides the Poisson PP $\Pi_\lambda \in \mathcal{P}_\lambda^\infty$ with PGF $G_{\Pi_\lambda}[v] = \exp\{\lambda \int_{\mathbb{R}^d}(v(x) - 1)\,\mathrm{d}x\}$ (i.e. $\gamma_{\Pi_\lambda}^{(k)} \equiv 0$ for $k \geq 2$) there are quite a few classes of Brillinger-mixing PPes, see e.g. Biscio and Lavancier (2016), Heinrich and Schmidt (1985) and Heinrich (1988).

One of them is formed by *Poisson cluster PPes* $\Psi_{pc}(\cdot) := \sum_{x\in s(\Phi)} \Phi_s^{(x)}((\cdot) - x) \sim P_{pc}$ with the Poisson PP $\Phi \sim \Pi_{\lambda_c}$ of cluster centres x and the countable family $\Phi_s^{(x)}$ (independent of Φ) of independent copies of a generic ($\mathbf{P}$—a.s.) finite PP $\Psi_s \sim P_s$, see Chapt. 10.2 in Daley and Vere-Jones (2008). It turns out that $\gamma_{P_{pc}}^{(k)} \geq 0$ for $k \geq 2$ and $P_{pc} \in \mathcal{P}_\lambda^\infty$ with $\lambda = \lambda_c\,\mathbf{E}\Psi_s(\mathbb{R}^d)$ iff $\mathbf{E}\Psi_s^k(\mathbb{R}^d) < \infty$ for all $k \geq 2$ and the latter implies that $\Psi_{pc} \sim P_{pc}$ satisfies Cond(B). It is noteworthy that the diameter of $s(\Phi_s)$ has no explicit influence on the Brillinger-mixing property. For later purposes we give the PGF of Ψ_{pc}:

$$G_{P_{pc}}[v] = \exp\Big\{\lambda_c \int_{\mathbb{R}^d}\big(G_{P_s}[v((\cdot) + x)] - 1\big)\,\mathrm{d}x\Big\}\,. \tag{1.6}$$

Cond(B) expresses some kind of simultaneous asymptotic uncorrelatedness of the numbers $\Psi(B_1), \ldots, \Psi(B_k)$ when B_i and B_j are separated from each other by a large distance for $1 \leq i < j \leq k$ and any $k \geq 2$. As shown in the recently submitted paper (Heinrich, 2017) Cond(B) implies asymptotic mutual independence of these numbers provided the moment (or cumulant) measures of $\Psi \sim P$ determine P uniquely, see also Ivanoff (1982) for a stronger additional assumption. In other words, the additional assumption of determinacy of the moment problem for the PP $\Psi \sim P$ ensures that this PP is mixing (of any order), i.e. $P(T_x Y_1 \cap Y_2) \longrightarrow P(Y_1)\,P(Y_2)$ as $\|x\| \to \infty$ for any $Y_1, Y_2 \in \mathcal{N}$. For sufficient conditions put on the moment measures or their densities to yield a unique distribution P the reader is referred to Zessin (1983). To be complete, we define