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The moments of a diffusion process

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ABSTRACT

This paper investigates the moments of a diffusion process to derive the formulas for the nth exact moment. Instead of seeking the density or moment-generating functions, we utilize stochastic integrals and their properties to obtain the moments.

1. Introduction

In this paper we consider a diffusion process governed by the stochastic differential equation

$$dX_t = (\alpha X_t + \beta)dt + \sqrt{\gamma X_t^2 + \delta X_t + \zeta}\, dB_t \tag{1}$$

where B_t denotes a one-dimensional Brownian motion. This form of stochastic differential equations can represent the several well-known stochastic processes as its special cases. For instance, when γ and δ are zero, the diffusion process becomes the Ornstein–Uhlenbeck process. When γ and ζ are zero, it becomes the square-root process (Avellaneda and Laurence, 2000). This type of process finds various applications in financial modeling, including interest rate models, stochastic volatility models and time series models (Cox et al., 1985; Demni and Zani, 2009; Delbaen and Shirakawa, 2002; Barone-Adesi et al., 2005; Heston, 1993; Glasserman and Kim, 2011; Vasicek, 1977).

In typical situations, the density function or the moment-generating function of a random variable is first obtained or given. And then its moments are computed either by integrating a power function together with the density function or by differentiating the moment-generating function. As to diffusion processes, one can find the transition probability density functions by solving the corresponding Kolmogorov forward or Fokker–Planck equations for special cases (see Feller, 1951, Wong, 1964). One can also obtain the moment-generating function of the square-root process by using its characteristic function and the corresponding Ricatti equations (Avellaneda and Laurence, 2000; Jondeau et al., 2007). Even though the density or moment-generating functions can be found for some special cases, it would not be straightforward to compute the moments if one need to obtain high-order moments.

The main purpose of this paper is to investigate the moments of the diffusion process (1) and to derive the formulas for the moments. Instead of seeking the density or moment-generating functions, we utilize stochastic integrals and their properties to find the moments. These formulas provide a simple computation of the nth moment. This leads to a very effective means to validate the performance of numerical simulations in stochastic modeling when applying the diffusion process (1).

The rest of the paper is organized as follows. In Section 2 we provide the main results by deriving the formulas of the moments. In Section 3 a numerical experiment is performed to compute the moments of the diffusion process. Section 4 concludes the paper.

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2. Main results

Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Let $\mathbb{X} = \{X_t : t \geq 0\}$ be a diffusion process on $(\Omega, \mathcal{F}, \mathbb{P})$ with state space $\mathbb{R}$; that is, X_t takes values in $\mathbb{R}$ for all $t \geq 0$. We consider the diffusion process $\mathbb{X}$ as the solution of the stochastic differential equation

$$\begin{cases} dX_t = (\alpha X_t + \beta)dt + \sqrt{\gamma X_t^2 + \delta X_t + \zeta}\, dB_t \\ X_0 = x_0 \end{cases} \tag{2}$$

where $\mathbb{B} = \{B_t : t \geq 0\}$ is a one-dimensional Brownian motion. We assume that X_t is well-defined for all $t \geq 0$ so that negative values are not achieved inside the square-root sign. We let $\mu(x) = \alpha x + \beta$ and $\sigma(x) = \sqrt{\gamma x^2 + \delta x + \zeta}$. Then, we have $|\mu(x)| + |\sigma(x)| \leq K(1 + |x|)$ for some constant $K > 0$ so that there exists at least a unique weak solution of Eq. (2) (see Stroock and Varadhan, 1979).

Throughout this paper we assume that all the moments of X_t are finite. More precisely, we suppose $\mathbb{E}[X_t^n] < \infty$ for all nonnegative integer n. For instance, the diffusion process $\mathbb{X}$ becomes the Ornstein–Uhlenbeck process when γ and δ are zero. Thus, its nth moment is finite because the Ornstein–Uhlenbeck process is a Gaussian process. In addition, $\mathbb{X}$ becomes the square-root process when γ and ζ are zero, and the distribution of the square-root process follows a noncentral chi-squared distribution (see Avellaneda and Laurence, 2000, Cox et al., 1985). Hence, its nth moment is finite.

We define $m_n(t) \triangleq \mathbb{E}[X_t^n]$ to represent the nth moment of X_t for $n = 0, 1, 2, \cdots$, where $m_0(t) = \mathbb{E}[X_t^0] \equiv 1$. In what follows we consider a power function $f(x) = x^n$ and let $Y_t = f(X_t)$ so that Y_t is again an Ito process. Applying Ito's Lemma on Y_t, we have $dY_t = f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2$ which results in

$$dX_t^n = (a_n X_t^n + b_n X_t^{n-1} + c_n X_t^{n-2})dt + nX_t^{n-1}\sqrt{\gamma X_t^2 + \delta X_t + \zeta}\, dB_t \tag{3}$$

where a_n, b_n and c_n are defined by

$$a_n \triangleq n\alpha + \frac{1}{2}n(n-1)\gamma, \;\; b_n \triangleq n\beta + \frac{1}{2}n(n-1)\delta, \;\; c_n \triangleq \frac{1}{2}n(n-1)\zeta.$$

Here we set $a_0 \equiv 0$, $b_0 \equiv 0$ and $c_0 \equiv 0$. Eq. (3) can be written in integral form

$$X_t^n = x_0^n + \int_0^t (a_n X_s^n + b_n X_s^{n-1} + c_n X_s^{n-2})ds + \int_0^t nX_s^{n-1}\sqrt{\gamma X_s^2 + \delta X_s + \zeta}\, dB_s. \tag{4}$$

To find the moments of the process X_t, we take expectation on both sides of (4) and differentiate with respect to t on each side so that we obtain the following differential equation: for $n \geq 1$

$$\begin{cases} m_n'(t) = a_n m_n(t) + b_n m_{n-1}(t) + c_n m_{n-2}(t) \\ m_n(0) = x_0^n \end{cases} \tag{5}$$

with $m_0(t) \equiv 1$ and $m_{-1}(t) \equiv 0$.

Theorem 1. *Suppose that the elements of the set $\{a_0, a_1, \cdots, a_N\}$ are distinct. Then the solution of (5) for $n = 0, \cdots, N$ is*

$$m_n(t) = \sum_{i=0}^{n} \xi_{ni} \exp(a_i t) \tag{6}$$

where ξ_{ni} satisfies the following recurrence relations: for $0 \leq i \leq n-2$

$$\xi_{ni} = \frac{b_n \xi_{n-1,i} + c_n \xi_{n-2,i}}{a_i - a_n}, \quad \xi_{n,n-1} = \frac{b_n \xi_{n-1,n-1}}{a_{n-1} - a_n}, \quad \xi_{nn} = x_0^n - \sum_{i=0}^{n-1} \xi_{ni}$$

with $\xi_{0,0} = 1$, $\xi_{1,1} = (x_0 + \frac{b_1}{a_1})$ and $\xi_{1,0} = -\frac{b_1}{a_1}$.

Proof. We start with $m_0(t) \equiv 1$. Given the preceding moments, the analytic solution to (5) is

$$m_n(t) = \exp(a_n t)\left(x_0^n + \int_0^t q_n(s)\exp(-a_n s)ds\right) \tag{7}$$

with $q_n(t) \triangleq b_n m_{n-1}(t) + c_n m_{n-2}(t)$. For instance, $q_n(t)$ is equal to b_1 when $n = 1$ so that we have $m_1(t) = (x_0 + \frac{b_1}{a_1})e^{a_1 t} - \frac{b_1}{a_1}e^{a_0 t}$ with $a_0 \equiv 0$. For $n = 2$, we have $q_2(t) = b_2 m_1(t) + c_2$, and it is straightforward to find the definite integral of $q_2(t)\exp(-a_2 t)$ from zero to t. This leads to the analytic solution for $m_2(t)$ that is a linear combination of $e^{a_0 t}$, $e^{a_1 t}$ and $e^{a_2 t}$. Suppose that $m_k(t)$ is a linear combination of $e^{a_0 t}, e^{a_1 t}, \cdots, e^{a_k t}$ for $k = 0, 1, \cdots, n-1$. Then $q_n(t)$ is also a linear combination of $e^{a_0 t}, e^{a_1 t}, \cdots, e^{a_{n-1} t}$. Because the elements of the set $\{a_0, a_1, \cdots, a_N\}$ are distinct, Eq. (7) implies that $m_n(t)$ should be a