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# Large deviations for some logarithmic means in the case of random variables with thin tails☆

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ABSTRACT

In this paper we generalize Theorem 3.3 in Giuliano and Macci (2011). More precisely we prove the full large deviation principle without assuming a particular condition in that theorem and, moreover, we give some results for the case of random variables with thin tails (and not super-exponential tails). As an application we deduce some consequences for the logarithmic means of some random variables expressed in terms of a $C$-process.




## 1. Introduction

The theory of large deviations gives an asymptotic computation of small probabilities on an exponential scale (see e.g. Dembo and Zeitouni, 1998 as a classical reference on this topic). Some results on large deviations in the literature concern the *Almost Sure Central Limit Theorem* (ASCLT for short), namely the almost sure weak convergence to the standard Normal distribution of the sequences of random measures

$$\left\{\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}1_{\{X_k\in\cdot\}} : n\geq 2\right\} \tag{1}$$

where $X_k := \frac{U_1+\cdots+U_k}{\sqrt{k}}$, and $\{U_n : n \geq 1\}$ is a sequence of i.i.d. centered random variables with unit variance. Of course we also have

$$\left\{\frac{1}{L(n)}\sum_{k=1}^{n}\frac{1}{k}1_{\{X_k\in\cdot\}} : n\geq 1\right\}, \text{ where } L(n) := \sum_{k=1}^{n}\frac{1}{k}. \tag{2}$$

The ASCLT was proved independently in Brosamler (1988), Fisher (1987) and Schatte (1988) under strong moment assumptions; successive refinements appear in Fisher (1989) and Lacey and Philipp (1990), in which only finite variance

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is required. Some generalizations and/or some other results based on different almost sure weak convergence results (with possibly different limits) can be found in Berkes and Csaki (2001), Fahrner and Stadtmüller (1998), Cheng et al. (1998), Fahrner (2000) and Hörmann (2007).

Large deviation results for the ASCLT are Theorem 1 in March and Seppäläinen (1997) (the expression of the rate function is provided by Theorem 3 in the same reference) for the sequence in (1) and Theorem 1.1 in Heck (1998) for the sequence in (2). Both results are proved assuming that all the (common) moments of the random variables $\{U_n : n \geq 1\}$ are finite; the optimality of the moment assumptions is discussed in Lifshits and Stankevich (2001). We also recall Rouault et al. (2002) which provides the large deviation principle for the so called Lévy strong arc-sine law (see Lévy, 1939).

In this paper we do not consider sequences of random measures but sequences of random variables. We typically deal with a sequence of random variables $\{Z_n : n \geq 2\}$ defined by

$$Z_n := \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} X_k,$$

where $\{X_n : n \geq 1\}$ is a sequence of random variables which converges weakly to a centered Normal distribution. Large deviation results for this kind of sequences can be found in Giuliano and Macci (2011) and Giuliano and Macci (2015); actually in those references the weak limit of the sequence $\{X_n : n \geq 1\}$ could not be Gaussian and, moreover, the results in Giuliano and Macci (2015) concern a generalization of logarithmic means.

The main result in this paper is Theorem 3.1, which gives some generalizations and refinements of Theorem 3.3 in Giuliano and Macci (2011). That theorem provides the full large deviation principle for the sequence of random variables $\{Z_n : n \geq 2\}$ defined by

$$Z_n := \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k\sqrt{k}} \sum_{i=1}^{k} U_i,$$

where $\{U_n : n \geq 1\}$ is a sequence of i.i.d. centered random variables with finite variance $\sigma^2 > 0$ which satisfy suitable conditions; in particular the tail of the random variables $\{U_n : n \geq 1\}$ is super-exponential. The new contributions of Theorem 3.1 are the following:

- we do not require condition **(C2)** used in Theorem 3.3 in Giuliano and Macci (2011);
- we can prove large deviation results when the random variables $\{U_n : n \geq 1\}$ have thin tails; unfortunately we can prove a full large deviation principle only when the tails are super-exponential (as required in Theorem 3.3 in Giuliano and Macci, 2011).

For the sake of completeness we present some examples of distributions with thin tails, that are not super-exponential tails. It is worth noting that some of them, in the domain of "perpetuities", have interest in some applications (see e.g. Goldie and Grubel, 1996 for details).

In the final part of the paper we present an application of Theorem 3.1 to the $C$-processes introduced in Williams (1973) (see also Fang and Wu, 2016 as a reference with large deviation results for $C$-processes).

We conclude with the outline of the paper. We start with some preliminaries on large deviations in Section 2. In Section 3 we prove Theorem 3.1, and we present some examples. Finally Section 4 is devoted to the application of Theorem 3.1 to the $C$-processes.

## 2. Preliminaries on large deviations

We refer to Dembo and Zeitouni (1998) (pages 4–5). Let $\mathcal{X}$ be a topological space equipped with its completed Borel $\sigma$-field. A sequence of $\mathcal{X}$-valued random variables $\{Z_n : n \geq 1\}$ satisfies the large deviation principle (LDP for short) with speed function $v_n$ and rate function $I$ if: $\lim_{n\to\infty} v_n = \infty$; $I : \mathcal{X} \to [0, \infty]$ is a lower semi-continuous function;

$$\limsup_{n\to\infty} \frac{1}{v_n} \log P(Z_n \in F) \leq -\inf_{x\in F} I(x) \text{ for all closed sets } F;$$

$$\liminf_{n\to\infty} \frac{1}{v_n} \log P(Z_n \in G) \geq -\inf_{x\in G} I(x) \text{ for all open sets } G.$$

A rate function $I$ is said to be *good* if its level sets $\{\{x \in \mathcal{X} : I(x) \leq \eta\} : \eta \geq 0\}$ are compact.

In this paper we apply the Gärtner–Ellis theorem (see e.g. Theorem 2.3.6 in Dembo and Zeitouni, 1998) and, more precisely, its version on $\mathbb{R}$. In view of this we recall some preliminaries (see e.g. Assumption 2.3.2 and Definitions 2.3.3–2.3.5 in Dembo and Zeitouni, 1998).

**Assumption 2.1.** For all $\theta \in \mathbb{R}$,

$$\Lambda(\theta) := \lim_{n\to\infty} \frac{1}{v_n} \log \mathbb{E}[e^{v_n \theta Z_n}] \tag{3}$$

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