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A note on exponential stability of non-autonomous linear stochastic differential delay equations driven by a fractional Brownian motion with Hurst index $> \frac{1}{2}$

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ABSTRACT

We prove a criterion for the almost sure exponential stability of the scalar non-autonomous linear stochastic differential delay equations driven by a fractional Brownian motion with Hurst index $> 1/2$. To do that, we need a version of existence and uniqueness of the solution for non-autonomous linear system.

1. Introduction

Consider the following non-autonomous linear stochastic differential delay equations driven by a fractional Brownian motion (fSDDE)

$$dx(t) = [A(t)x(t) + B(t)x(t-r)]\,dt + [C(t)x(t) + D(t)x(t-r)]\,dB^H(t),\ x_0 \in C_r, \tag{1.1}$$

where $A,\ B,\ C,\ D$ are continuous matrix functions from $\mathbb{R}_+$ to $\mathbb{R}^{d\times d}$, r is a constant delay, $B^H(t)$ is a fractional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the Hurst index $H \in (1/2, 1)$ (see Duc et al., 2015) and $C_r := C([-r, 0], \mathbb{R}^d)$ is the space of all continuous functions from $[-r, 0]$ to $\mathbb{R}^d$ endowed by the sup norm $\|.\|_\infty$. Following the notations in Boufoussi and Hajji (2011), define

$$f(t, \xi) = A(t)\xi(0) + B(t)\xi(-r) \ \text{ and } \ g(t, \xi) = C(t)\xi(0) + D(t)\xi(-r), \tag{1.2}$$

where $t \in \mathbb{R}_+$; $\xi \in C_r$ and $x_t \in C_r$ defined by $x_t(u) = x(t+u)$, for all $u \in [-r, 0]$.

Eq. (1.1) can be written in the integral form as

$$x(t) = \eta(0) + \int_0^t f(s, x_s)ds + \int_0^t g(s, x_s)dB^H(s),\ t \geq 0 \tag{1.3}$$
$$x_0 = \eta \in C_r,$$

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in which $\int_0^t g(s,x_s)dB^H(s)$ is understood as a path wise integral which can be represented by fractional derivatives, see e.g. Nualart and Răşcanu (2002), Young (1936) and Zähle (1998). Specifically, for each fixed $a,b\in\mathbb{R},\ a<b$, denote $W_a^{\alpha,1}([a,b],\mathbb{R}^d)$ is the space of integrable functions $h:[a,b]\longrightarrow\mathbb{R}^d$ such that

$$\|h\|_{\alpha,1}:=\int_a^b\left(\frac{|h(s)|}{(s-a)^\alpha}+\int_a^s\frac{|h(s)-h(u)|}{(s-u)^{1+\alpha}}du\right)ds<+\infty.$$

For $h\in W_a^{\alpha,1}([a,b],\mathbb{R}^d)$, define the left-sided fractional derivative $D_{a+}^\alpha h(u)$ by

$$D_{a+}^\alpha h(u)=\frac{1}{\Gamma(1-\alpha)}\left[\frac{h(u)}{(u-a)^\alpha}+\alpha\int_a^u\frac{h(u)-h(y)}{(u-y)^{1+\alpha}}dy\right]1_{(a,b)}(u).$$

Also denote by $W_b^{1-\alpha,\infty}([a,b],\mathbb{R}^d)$ the space of continuous functions $k:[a,b]\longrightarrow\mathbb{R}^d$ satisfying

$$\|k\|_{1-\alpha,\infty}:=\sup_{a\leqslant s<t\leqslant b}\frac{|k(t)-k(s)|}{(t-s)^{1-\alpha}}+\int_s^t\frac{|k(u)-k(s)|}{(u-s)^{2-\alpha}}du<+\infty.$$

Then for any $k\in W_T^{1-\alpha,\infty}([0,T],\mathbb{R}^d)$, introduce $k_{t-}(u):=k(u)-k(t)$ and define the right-sided fractional derivative $D_{t-}^{1-\alpha}k_{t-}(u)$ by

$$D_{t-}^{1-\alpha}k_{t-}(u)=\frac{(-1)^{1-\alpha}}{\Gamma(\alpha)}\left[\frac{k(u)-k(t)}{(t-u)^{1-\alpha}}+(1-\alpha)\int_u^t\frac{k(u)-k(y)}{(y-u)^{2-\alpha}}dy\right]1_{(a,t)}(u).$$

For $a<b$, denote by $C^\kappa([a,b],\mathbb{R}^d)$ the space of κ-Hölder functions from $[a,b]$ to $\mathbb{R}^d$, equipped with the norm

$$\|f\|_{\infty,\kappa,[a,b]}:=\|f\|_{\infty,[a,b]}+\|f\|_{\kappa,[a,b]},$$

where

$$\|f\|_{\infty,[a,b]}:=\sup_{u\in[a,b]}|f(u)|\text{ and }\|f\|_{\kappa,[a,b]}:=\sup_{a\leqslant u<v\leqslant b}\frac{|f(v)-f(u)|}{|v-u|^\kappa}.$$

It can be seen in Nualart and Răşcanu (2002) that

$$C^{\alpha+\varepsilon}([a,b],\mathbb{R}^d)\subset W_a^{\alpha,1}([a,b],\mathbb{R}^d)\text{ for any }\varepsilon>0$$

and

$$C^{1-\alpha+\varepsilon}([a,b],\mathbb{R}^d)\subset W_b^{1-\alpha,\infty}([a,b],\mathbb{R}^d)\subset C^{1-\alpha}([a,b],\mathbb{R}^d).$$

If $h\in C^\lambda([a,b],\mathbb{R}^d)$ and $\omega\in C^\nu([a,b],\mathbb{R})$ with $\lambda+\nu>1$, we have $h\in W_a^{\alpha,1}([a,b],\mathbb{R}^d)$ and $\omega\in W_b^{1-\alpha,\infty}([a,b],\mathbb{R})$ for each α such that $\lambda>\alpha>1-\nu$. After that, one can define the integral in the Zähle sense (Zähle, 1998) by

$$G(h)(t)=\int_a^t h(u)d\omega(u):=(-1)^\alpha\int_a^t D_{a+}^\alpha h(u)D_{t-}^{1-\alpha}\omega_{t-}(u)du,\ \ t\in(a,b). \tag{1.4}$$

It is well known (Nualart and Răşcanu, 2002) that

$$\left|\int_a^t h(u)d\omega(u)\right|\leqslant\Lambda_\alpha(\omega)\|h\|_{\alpha,1}, \tag{1.5}$$

where

$$\Lambda_\alpha(\omega):=\frac{1}{\Gamma(1-\alpha)}\sup_{a\leqslant s<t\leqslant b}|D_{t-}^{1-\alpha}\omega_{t-}(s)|\leqslant\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\|\omega\|_{1-\alpha,\infty}<+\infty.$$

For any fixed $T>0$, since Kolmogorov–Centsov's lemma (Nourdin, 2012, p. 8), B^H has almost all the sample paths in $C^\nu([0,T],\mathbb{R})$ for $\frac12<\nu<H$. Therefore one can write this equation in the deterministic form

$$x(t)=\eta(0)+\int_0^t f(s,x_s)ds+\int_0^t g(s,x_s)d\omega(s),\ 0\le t\le T, \qquad x_0=\eta\in C_r, \tag{1.6}$$

where $\omega(\cdot)\in C^\nu([0,T],\mathbb{R})$, f and g are defined as in (1.2), and the second integral is in the sense of (1.4). A $\mathbb{R}^d$-valued continuous function x is said to be a solution of Eq. (1.6) on $[-r,T]$ if it satisfies the equation and $x_0=x|_{[-r,0]}=\eta$ (see Hale and Verduyn Lunel, 1993).

Definition 1. A stochastic process $X:[-r,T]\times\Omega\to\mathbb{R}^d$ is said to be the solution of (1.3) if there exists $\Omega'\in\mathcal{F}$ such that $P(\Omega')=1$ and for all $\omega\in\Omega'$, $X(\cdot,\omega)$ is a solution of (1.6).

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