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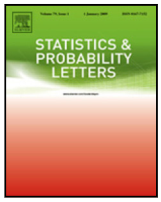

On limiting theorems for conditional causation probabilities of multiple-run-rules

Hsing-Ming Chang [a], Yung-Ming Chang [b,*], Winnie H.W. Fu [c], Wan-Chen Lee [**]

[a] *Department of Statistics, National Cheng Kung University, Tainan, Taiwan, ROC*
[b] *Department of Applied Mathematics, National Taitung University, Taitung, Taiwan, ROC*
[c] *Department of Mechanical Engineering, West Virginia University Institute of Technology, Beckley, USA*

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A B S T R A C T

Causation probabilities of multiple-run-rules based on patterns of multi-state trials have been used and studied in various fields such as quality control, reliability of engineering system, biology, DNA sequence analysis and survival analysis. In this manuscript, we derive the limiting results for two types of conditional causation probabilities for multiple-run-rules by using the finite Markov chain imbedding technique. Extension and numerical examples are given to illustrate the theoretical results.

1. Introduction

Multiple-run-rules $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_d)$ based multi-state trials $\{X_t\}_{t=1}^n$ have been used successfully in various scientific fields, such as quality control, reliability systems, biology, DNA sequence analysis, risk management, and survival analysis. It becomes an indispensable tool especially in quality control for making higher quality products and more reliably (Bersimis et al., 2014). The well-known Western Electric quality control scheme is a typical example, see Montgomery (2001).

The waiting time distribution of multiple-run-rules has been studied by many researchers especially when $\{X_t\}_{t=1}^n$ is a sequence of first-order Markov dependent trials. An illustration of the technique is seen in Fu et al. (2016) on the study of unconditional causation probabilities for a specified simple pattern or a group of simple patterns of multiple-run-rules for both independent and identically distributed (i.i.d.) and Markov dependent trials. In practice, researchers are often interested in analyzing the remaining useful life or the survival time of a system. In this manuscript, we mainly study the long term behavior of two types of conditional probabilities relating to system reliability based on multiple-run-rules for a sequence of i.i.d. and Markov dependent trials. In order to achieve this goal, the finite Markov chain imbedding (FMCI) technique and results of asymptotic distribution of runs and patterns developed in Fu and Johnson (2009) and Johnson and Fu (2014) will be used. The FMCI technique has been used in studying the reliability of various engineering systems and quality control systems (see, e.g., Koutras et al., 2007), a review in developments on this technique can be found in Cui et al. (2010). A simple numerical example will be given to illustrate our theoretical results.

2. Main results

Let $\{X_t\}_{t=1}^n$ be a sequence of independently and identically distributed (i.i.d.) or Markov dependent m-state random variables defined on a set $\mathcal{S} = \{a_1, \ldots, a_m\}$ of m symbols.

* Corresponding author.
** Correspondence to: 90 Elgin St Ottawa, Ontario Canada K1P 5E9.
E-mail addresses: eddchang@nttu.edu.tw (Y.-M. Chang), ophelia0825@gmail.com (W.-C. Lee).

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Definition 1. We say that Λ is a *simple pattern* of length k if it is composed of a specified sequence of k symbols (k is fixed) in $\mathcal{S}$, i.e., $\Lambda = x_1 x_2 \cdots x_k$, $x_i \in \mathcal{S}$ for $i = 1, \ldots, k$.

Let Λ_1 and Λ_2 be two simple patterns of lengths k_1 and k_2, respectively. We say that Λ_1 and Λ_2 are *distinct* if neither one is a segment of the other. Define the union of two simple patterns $\Lambda_1 \cup \Lambda_2$ to be the occurrence of either the pattern Λ_1 or the pattern Λ_2.

Definition 2. $\Lambda = \cup_{i=1}^{k} \Lambda_i$ is called a *compound pattern* if it is a union of k ($k \geq 2$) distinct simple patterns Λ_i, $i = 1, \ldots, k$. Note that the lengths of the simple patterns do not have to be the same.

Definition 3. Define the *waiting time* of the occurrence of a simple or compound pattern Λ as

$$W(\Lambda) = \min\left\{ n : \begin{array}{c} \text{the pattern } \Lambda \text{ has occurred} \\ \text{in the sequence } \{X_t\}_{t=1}^{n} \end{array} \right\}.$$

Let $\varphi_i = \varphi(\Lambda_i)$ be a decision rule based on the occurrence of a pattern Λ_i (simple or compound) in the sequence $\{X_t\}_{t=1}^{n}$. It has been shown in Fu et al. (2016) that the reliability of a system with multiple-run-rules $(\varphi_1, \ldots, \varphi_d)$ can be reduced to a system with a single run-rule $\varphi(\Lambda)$ based on the compound pattern $\Lambda = \cup_{i=1}^{k} \Lambda_i$. Let $W(\Lambda_i)$ be the waiting time for the occurrence of the simple pattern Λ_i, for $i = 1, \ldots, k$. Mathematically, we define waiting time $W(\Lambda)$ for the occurrence of the compound pattern $\Lambda = \cup_{i=1}^{k} \Lambda_i$ to be

$$W(\Lambda) = \min(W(\Lambda_1), \ldots, W(\Lambda_k)).$$

Given Λ, it is known that the random variable $W(\Lambda)$ is finite Markov chain imbeddable. The homogeneous imbedded Markov chain $\{Y_t\}_{t=0}^{\infty}$ defined on an arranged state space Ω has a transition probability matrix $\boldsymbol{M}$ composed of

$$\boldsymbol{M} = \begin{array}{c} \\ \Omega \setminus \mathcal{A} \\ \mathcal{A} \end{array} \begin{array}{c} \begin{array}{cc} \Omega \setminus \mathcal{A} & \mathcal{A} \end{array} \\ \left[\begin{array}{c:c} \boldsymbol{N} & \boldsymbol{C} \\ \hdashline \boldsymbol{O} & \boldsymbol{I} \end{array} \right] \end{array},$$

where $\mathcal{A} = \{\alpha_1, \ldots, \alpha_k\}$ is the set of all absorbing states corresponding to simple patterns $\Lambda_1, \ldots, \Lambda_k$, respectively, $\boldsymbol{N}$ is the essential transition probability matrix, each row of the matrix $\boldsymbol{C}$ is composed of transition probabilities going from a state in $\Omega \setminus \mathcal{A}$ to the absorbing states in $\mathcal{A}$, $\boldsymbol{O}$ is a zero matrix, and $\boldsymbol{I}$ is an identity matrix of size k.

The failure probability of a system is known to be

$$P(W(\Lambda) = n) = \boldsymbol{\xi}_0 \boldsymbol{N}^{n-1} (\boldsymbol{I} - \boldsymbol{N}) \boldsymbol{1}', \tag{1}$$

where $\boldsymbol{1} = (1, 1, \ldots, 1)$ is a row vector and $\boldsymbol{\xi}_0 = (1, 0, \ldots, 0)$ is a row vector as the initial state distribution of Y_0. Furthermore, for any $\alpha_i \in \mathcal{A}$, the unconditional causation probability is

$$P(W(\Lambda) = n \text{ and } W(\Lambda_i) = n) = \boldsymbol{\xi}_0 \boldsymbol{N}^{n-1} \boldsymbol{c}_i, \tag{2}$$

where $\boldsymbol{c}_i$ is the ith column vector in $\boldsymbol{C}$.

We define two conditional causation probabilities of interest as

$$\begin{aligned} \gamma_n(\alpha_i) &= P(W(\Lambda_i) = n | W(\Lambda) = n) \\ &= \frac{P(W(\Lambda) = n \text{ and } W(\Lambda_i) = n)}{P(W(\Lambda) = n)} \end{aligned} \tag{3}$$

and

$$\begin{aligned} \theta_n(\alpha_i) &= P(W(\Lambda_i) = n | W(\Lambda) > n - 1) \\ &= \frac{P(W(\Lambda) = n \text{ and } W(\Lambda_i) = n)}{P(W(\Lambda) > n - 1)}. \end{aligned} \tag{4}$$

The aspect of 'causation' in (2) is characterized by the probability of system failure at time n due to the occurrence of Λ based solely on the simple pattern Λ_i. We call (3) the conditional cause of failure probability for the absorbing state α_i and (4) the hazard probability for the absorbing state α_i that denotes the conditional probability of system failure at time n due to the occurrence of Λ based solely on the simple pattern Λ_i given that the system has been working at least till time $n - 1$. The following two theorems show the ergodicity of the conditional cause of failure probability and the hazard probability.

Let $1 > \lambda_1 > |\lambda_2| \geq |\lambda_3| \cdots \geq |\lambda_\ell|$ be eigenvalues of the essential matrix $\boldsymbol{N}$ with corresponding normalized right-hand side orthogonal column eigenvectors $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \ldots, \boldsymbol{\eta}_\ell$ (such that $\boldsymbol{N}\boldsymbol{\eta}_i = \lambda_i \boldsymbol{\eta}_i$ for $i = 1, \ldots, \ell$). In order to simplify our proofs, we assume without loss of generality that the algebraic multiplicity of the largest eigenvalue λ_1 is one. To prove our main results, we first prove Lemma 1 by using the result of Fu and Johnson (2009) on approximating the tail probabilities.

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