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# A new result on lifetime estimation based on skew-Wiener degradation model

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ABSTRACT

This paper explores the influence of measurement errors on lifetime estimation based on Wiener degradation model with the skew normal distribution. The exact and explicit expressions of the lifetime distribution are derived, which subsumes several existing results as special cases.



## 1. Introduction

The use of degradation data to estimate the lifetime of highly reliable products has become a focus in reliability engineering and condition based maintenance (Lawless, 2002; Ye et al., 2012; Li et al., 2016; Pan et al., 2016). The choice of the degradation model plays a crucial role in lifetime estimation (Si et al., 2013a; Peng and Tseng, 2009; Tsai et al., 2011). Since the degradation of many products often evolves in a stochastic way such as rotating machines and batteries, the stochastic processes are well suitable to model the degradation process of a product and estimate its lifetime information. The Wiener processes have been widely used to model the non-monotonic degradation process, which have some favorable mathematical properties and physical interpretations (Wang, 2010; Pan et al., 2017; Jin et al., 2013). The observed degradation data are often contaminated by measurement errors (ME) due to imperfect instruments and random environments (Whitmore, 1995; Zhai et al., 2016; Zhai and Ye, 2017; Wen et al., 2017). Therefore, it is necessary to consider the effect of ME on the lifetime estimation performance. A well-adopted degradation model based on Wiener processes with ME can be expressed as (Ye et al., 2013)

$$Y(t) = X(t) + \varepsilon = \beta\Lambda(t) + \sigma W(\Lambda(t)) + \varepsilon, \tag{1}$$

where $\beta, \sigma$ and $\Lambda(t)$ are the drift parameter, diffusion parameter and transformed time scale respectively. $X(t)$ represents the underlying degradation process. $W(\Lambda(t))$ represents a standard Brown motion process when $\Lambda(t) = t$. The measurement error $\varepsilon$ is assumed to be normally distributed with variance $\sigma_\varepsilon^2$ and zero mean, which is generally supposed to be s-independent with $\beta$.

The degradation rates among different individuals generally have a great deal of difference owing to the randomness of raw materials or manufacturing processes in engineering practice. As such, the unit-to-unit variability should be integrated

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into the degradation model to obtain more accurate lifetime estimation results. In the existing Wiener-process-based degradation models, the unit-to-unit variability is characterized by assuming the drift parameter as the normal random variable, and the diffusion parameter is regarded as a constant for describing the degradation characteristic common to all individuals in the population. The related works of imposing a normal distribution on the drift parameter can be found in Ye et al. (2013), Si et al. (2013b), Wang et al. (2014) and the references therein. However, the degradation rates among a batch of products can manifest an asymmetric behavior in practical application. In this case, the normality assumption fails to fit degradation data of such products due to the symmetry of the normal distribution. Therefore, it is necessary to relax the normality assumption to provide more reliable lifetime information for preventive maintenance scheduling.

Recently, Peng and Tseng (2013) presented a lifetime estimation approach based on the skew-Wiener linear degradation model. To capture a broad range of non-normal, asymmetric behavior in the unit-to-unit variability among a batch of products, they assumed the drift parameter follows the skew normal distribution instead of the normal distribution, and obtained a mathematically tractable degradation model. However, it is worth noting that the effect of ME on the life distribution is not taken into account in their method, therefore the obtained life estimation results are approximations. Driven by this work, the objective of this paper is to investigate the influence of ME on lifetime estimation in terms of the skew-Wiener degradation model. By simultaneously considering the unit-to-unit variability and ME, the closed-form lifetime distribution is obtained based on the concept of the first hitting time (FHT), and the proposed lifetime estimation results include several existing results as special cases.

The rest of this paper is organized as follows. Section 2 presents the closed-form expressions of the lifetime distribution in terms of the skew-Wiener degradation model with ME. Section 3 provides a numerical example to illustrate the effectiveness of the proposed results. Section 4 concludes this paper.

## 2. Derivation of the life distribution

In this section, we will illustrate how to drive the lifetime distribution in terms of the degradation model (1). In this paper, the skew normal distribution is used to represent the unit-to-unit variability by drawing on the idea of Peng and Tseng (2013). To be specific, $\beta$ follows the skew normal distribution denoted by $\beta \sim SN(\mu_\beta, \sigma_\beta^2, \alpha)$ if the probability density function (PDF) is given by

$$f(\beta) = \frac{2}{\sigma_\beta}\phi\left(\frac{\beta-\mu_\beta}{\sigma_\beta}\right)\Phi\left(\alpha\frac{\beta-\mu_\beta}{\sigma_\beta}\right),\ \beta, \alpha \in R, \sigma_\beta > 0 \tag{2}$$

where $\mu_\beta$ is the location parameter, $\sigma_\beta$ is the scale parameter, $\alpha$ is the shape parameter. $\phi(\cdot)$ and $\Phi(\cdot)$ represent the PDF and cumulative distribution function (CDF) of the standard normal distribution, respectively. It is noteworthy that the skew normal distribution subsumes several common distributions as limiting cases. For instance, the skew normal distribution is reduced to the normal distribution when $\alpha = 0$. If $\alpha \to +\infty$, the skew normal distribution comes down to the half-normal distribution $HN(\mu_\beta, \sigma_\beta^2)$ with the following PDF

$$f(\beta) = \begin{cases} \frac{2}{\sigma_\beta}\phi\left(\frac{\beta-\mu_\beta}{\sigma_\beta}\right), & \beta \geq \mu_\beta, \\ 0, & \beta < \mu_\beta. \end{cases} \tag{3}$$

A product is generally considered as a failure when the degradation path first reaches a predefined failure threshold $\omega$ (Peng and Tseng, 2009). For this reason, the concept of the FHT is often used to define the lifetime. Without loss of generality, we assume the degradation path increases over time. For the degradation process without ME, based on the FHT concept, the lifetime $T$ can be defined as

$$T = \inf\{t : X(t) \geq \omega | X(0) < \omega\}. \tag{4}$$

As for the degradation process with ME, the failure threshold $\omega$ is often defined by the industrial standard, which depends on the knowledge and experience of the expert by using the observed degradation data (Si et al., 2013a). In this case, the lifetime $T_e$ is defined as

$$T_e = \inf\{t : Y(t) \geq \omega | Y(0) < \omega\} = \inf\{t : X(t) \geq \omega_e | X(0) < \omega_e\}, \tag{5}$$

where $\omega_e = \omega - \varepsilon$ is normally distributed with variance $\sigma_\varepsilon^2$ and mean $\omega$. It becomes obvious that the lifetime $T_e$ is calculated by the time of $Y(t)$ first hitting the threshold $\omega$, which is different from the lifetime $T$. In general, $X(t)$ cannot be observed directly because perfect measurement of the degradation process is impossible. Therefore, $Y(t)$ is used to define the lifetime, which is greatly different from the work of Peng and Tseng (2013). In fact, the definition (5) can consider the effect of ME on the lifetime in comparison with the definition (4).

If $\omega_e$ and $\beta$ are constant, based on the property of the Wiener process, $\Lambda(T_e)$ follows the inverse Gaussian distribution (Whitmore and Schenkelberg, 1997). Therefore, the CDF of $T_e$ can be given by (Ye et al., 2013)

$$F_{T_e|\beta,\omega_e}(t) = \Phi\left(\frac{\beta\Lambda(t)-\omega_e}{\sigma\sqrt{\Lambda(t)}}\right) + \exp\left(\frac{2\beta\omega_e}{\sigma^2}\right)\Phi\left(\frac{-\omega_e-\beta\Lambda(t)}{\sigma\sqrt{\Lambda(t)}}\right). \tag{6}$$