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On covariance functions with slowly or regularly varying modulu of continuity

J.M.P. Albin [1]

Department of Mathematical Sciences, Chalmers University of Technology, 412 96 Gothenburg, Sweden

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ABSTRACT

By means of Fourier transforms we show that more or less any regularly varying or slowly varying function can feature as the modulu of continuity in squared mean sense of a stationary stochastic process.

1. Introduction

In another manuscript we are working with extreme value theory in the sense of e.g., Pickands (1969a, b) and Leadbetter et al. (1983, Chapter 12) for stationary Gaussian processes $\{\xi(t)\}_{t\in\mathbb{R}}$. Unlike their setting with a polynomial modulu of continuity in squared mean sense, which is to say that $r(t) = \mathbf{Cov}\{\xi(s), \xi(s+t)\}$ satisfies

$$\lim_{t\to 0}(r(0) - r(t))/|t|^{\alpha} = C$$

for some constants $C > 0$ and $\alpha \in (0, 2]$ we there study the little before studied case with a lower than polynomial modulu of continuity so that

$$\lim_{t\to 0}(r(0) - r(t))/|t|^{\alpha} = \infty$$

for all $\alpha > 0$. As we in the mentioned work instead impose conditions such as

$$\lim_{t\to 0}(r(0) - r(t))/(\log(1/|t|))^{-\beta} = C$$

E-mail address: palbin@chalmers.se.

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for some constants $C > 0$ and $\beta > 1$ (which is well-known to ensure continuity of sample paths), i.e., a slowly varying modulu of continuity, we were naturally led to the question what moduli of continuity can exist? And that is the topic of the present work.

Somewhat over simplified (but not much) we show that any slowly varying as well as any regularly varying with exponent less than or equal to 2 modulu of continuity can exist. In particular this shows that the above mentioned requirements are legitimate, i.e., there exist Gaussian processes with such covariance functions.

We also attend to the issue of what higher order terms can feature in the asymptotic behavior of a covariance function close to zero as (unlike in the case with polynomial moduli of continuity) second order terms turn out to be useful to get better estimates for extremes in the case of a slowly varying moduli of continuity.

Regularly varying (as a generalization or polynomial) moduli of continuity in extremes were first considered by Qualls and Watanabe (1972). Estimates for extreme value theory for Gaussian processes with poorer than polynomial moduli of continuity have been considered by Adler (1990) and Samorodnitsky (1991), but see also Berman (1989) for related but somewhat different results.

2. Preparations

We will first for the convenience of the reader list some facts about slowly varying functions that can be found in the book by Bingham et al. (1987): Given a constant $c > 0$ a measurable function $\ell : [c, \infty) \to (0, \infty)$ is called slowly varying at ∞ if

$$\lim_{x\to\infty} \ell(\lambda x)/\ell(x) = 1 \quad \text{for } \lambda > 0 \tag{2.1}$$

(Bingham et al., 1987, Eq. 1.2.1). Similarly, a measurable function $\ell : (0, c] \to (0, \infty)$ is called slowly varying at 0 if

$$\lim_{x\downarrow 0} \ell(\lambda x)/\ell(x) = 1 \quad \text{for } \lambda > 0.$$

Clearly, ℓ is slowly varying ∞ if and only if $\ell(1/\cdot)$ is slowly varying at 0. We will be interested in covariance functions $r : \mathbb{R} \to [-r(0), r(0)]$ such that $r(0) - r(t)$ is slowly varying at zero. As the literature on slow variation is written for slow variation at ∞ we will phrase our results for covariance functions r such that $r(0) - r(t) = \ell(1/t)$ with ℓ slowly varying at ∞. When we talk about slow variation henceforth we mean slow variation at ∞ unless something else is explicitly stated.

By the uniform convergence theorem (Bingham et al., 1987, Theorem 1.2.1) the convergence (2.1) must be uniform over each compact subset of λ-values in $(0, \infty)$.

Each slowly varying function is asymptotically equivalent to a $C^\infty([c, \infty))$ slowly varying function such that

$$\lim_{x\to\infty} x\,\ell'(x)/\ell(x) = 0 \tag{2.2}$$

(Bingham et al., 1987, Theorem 1.3.3). For $\ell(x)$ slowly varying and integrable over $[c, \infty)$ a special case of the so called Karamata's Theorem (Bingham et al., 1987, Proposition 1.5.9b) asserts that

$$\int_x^\infty \frac{\ell(y)}{y}\,dy \quad \text{is slowly varying with} \quad \lim_{x\to\infty} \frac{1}{\ell(x)} \int_x^\infty \frac{\ell(y)}{y}\,dy = \infty.$$

If $\ell(x)$ is absolutely continuous with $\lim_{x\to\infty} \ell(x) = 0$ and $-x\,\ell'(x)$ is slowly varying, then Karamata's Theorem shows that $\ell(x)$ is slowly varying and satisfies (2.2). For $\ell(x)$ $n-1$ times continuously differentiable with an absolutely continuous $n-1$th derivative we write $\ell^{(0)}(x) = \ell(x)$ and $\ell^{(k)}(x) = x\,\frac{d}{dx}\ell^{(k-1)}(x)$ for $k = 1, \ldots, n$. As above, if

$$(-1)^n \ell^{(n)}(x) \quad \text{is slowly varying and} \quad \lim_{x\to\infty} \ell^{(k)}(x) = 0 \quad \text{for } k = 0, \ldots, n-1, \tag{2.3}$$

then $(-1)^k \ell^{(k)}(x)$ are slowly varying for $k = 0, \ldots, n-1$ and satisfy (2.2), that is

$$\lim_{x\to\infty} \ell^{(k)}(x)/\ell^{(k-1)}(x) = 0 \quad \text{for } k = 1, \ldots, n.$$

Typically, an absolutely continuous slowly varying function $\ell(x)$ encountered in practice will have $x\,\ell'(x)$ or $-x\,\ell'(x)$ slowly varying. In theory, things are more complicated. One important result in this direction is that if $\ell'(x)$ is ultimately monotone, then $x\,\ell'(x)$ or $-x\,\ell'(x)$ is slowly varying if and only if $\ell(x)$ belongs to the so called de Haan subclass Π of slowly varying functions (Bingham et al., 1987, Section 3.0 and Corollary 3.6.9).

For ℓ slowly varying and bounded away from 0 and ∞ on compact subsets of $[c, \infty)$ Potter's Theorem (Bingham et al., 1987, Theorem 1.5.6) asserts that, given any constant $\delta > 0$,

$$\frac{1}{A}(y/x)^{-\delta} \le \frac{\ell(y)}{\ell(x)} \le A\,(y/x)^{\delta} \quad \text{for } y \ge x \ge c, \quad \text{for some constant } A = A(\delta) > 1.$$

If ℓ is not thus bounded from the beginning it is enough to modify it to be bounded on a suitable bounded interval $[c, X]$ to make Potter's Theorem valid. This is a technique that we will utilize without mentioning whenever it is needed henceforth as all statements we are proving only involve the asymptotic behavior of $\ell(x)$ as $x \to \infty$.

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