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Martingale transforms between Hardy–Lorentz spaces

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ABSTRACT

Using the technique of Burkholder's martingale transforms, the interchanging relations between two Hardy–Lorentz spaces of martingales are characterized. More precisely, it is proved that the elements in $H^s_{p_1,q}$ (resp. $H^s_{p_1,q_1}$) are none other than the martingale transforms of those in $H^s_{p_2,q}$ (resp. $H^s_{p_2,q_2}$), when $0 < p_1 < p_2 < \infty$, $0 < q < \infty$ (resp. $0 < p_1 < p_2 < \infty$, $0 < q_1 < q_2 < \infty$ and $\frac{p_1}{p_2} = \frac{q_1}{q_2}$), and a martingale is in $H^s_{p,q}$ for $1 < p, q < \infty$ if and only if it is a martingale transform of some martingale from BMO_2.

1. Introduction

The motivation in this paper comes from the classical results of Chao and Long (1992b, a), as well as the similar results of Weisz (1994a, b). The concept of martingale transforms was first introduced by Burkholder (1966). It is shown that the martingale transforms are especially useful to study the relations between the "predictable" Hardy spaces of martingales, such as H^s_p, which is associated with the conditional quadratic variation of martingales. The "characterization" of such spaces by martingale transforms was provided in Chao and Long (1992b, a): if $0 < p_1 < p_2 < \infty$ the elements in the space $H^s_{p_1}$ are none other than the martingale transforms of those in $H^s_{p_2}$; if $p_2 = \infty$ the space $H^s_{p_2}$ could be replaced by BMO_2. As for Hardy–Orlicz spaces, the corresponding conclusions were proved by Ishak and Mogyoródi (1982), Meng and Yu (2009), Yu (2011), Yu and Zhuang (2014) and Yu and Yin (2014), respectively.

Since 1951 when they were first introduced by G.G. Lorentz in Lorentz (1951), Lorentz spaces have attracted more and more attention. In particular, recently, the study of the martingale properties of Hardy–Lorentz spaces has become one of the hot topics and many important results have been obtained. Fan et al. (2010) discussed Hardy–Lorentz spaces' basic properties, embedding relations and interpolation spaces. Jiao et al. (2009b, a) studied the weak type inequalities for maximal operator and square operator on Hardy–Lorentz spaces. In Ren (2015) and Liu et al. (2017) the dual spaces of Hardy–Lorentz spaces are identified for real-valued and vector-valued martingales, respectively.

The main purpose of this paper is to extend some classical results of martingale transforms from Hardy spaces to Hardy–Lorentz spaces. More precisely, we are interested in the characterization by means of martingale transforms about the interchanging relations between two Hardy–Lorentz spaces of martingales $H^s_{p_1,q}$ and $H^s_{p_2,q}$, as well as $H^s_{p_1,q_1}$ and $H^s_{p_2,q_2}$ and that between $H^s_{p,q}$ and BMO_2.

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2. Preliminaries and notations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and f a measurable function defined on Ω. The decreasing rearrangement of f is the function f^* defined by

$$f^*(t) := \inf\{s > 0 : \mathbb{P}(|f| > s) \leq t\}, \quad t \geq 0.$$

The Lorentz space $L_{p,q}(\Omega) = L_{p,q}$, $0 < p < \infty$, $0 < q \leqslant \infty$, consists of measurable functions f with finite quasi-norm $\|f\|_{p,q}$ given by

$$\|f\|_{p,q} := \left(\frac{q}{p}\int_0^\infty [t^{1/p}f^*(t)]^q \frac{\mathrm{d}t}{t}\right)^{1/q}, \quad 0 < q < \infty,$$

$$\|f\|_{p,\infty} := \sup_{t>0} t^{1/p}f^*(t), \quad q = \infty.$$

It will be convenient for us to use an equivalent definition of $\|f\|_{p,q}$, namely

$$\|f\|_{p,q} = \left(q\int_0^\infty [t\mathbb{P}(|f(x)| > t)^{1/p}]^q \frac{\mathrm{d}t}{t}\right)^{1/q}, \quad 0 < q < \infty,$$

$$\|f\|_{p,\infty} = \sup_{t>0} t\mathbb{P}(|f(x)| > t)^{1/p}, \quad q = \infty.$$

Let $\{\mathcal{F}_n\}_{n\geqslant 0}$ be a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}=\sigma(\cup_{n\geqslant 0}\mathcal{F}_n)$. We denote the expectation operator and the conditional expectation operators relative to $\{\mathcal{F}_n\}_{n\geqslant 0}$ by $E(\cdot)$ and $E(\cdot|\mathcal{F}_n)$, respectively.

Let $f = \{f_n\}_{n\geqslant 0}$ be a martingale adapted to $\{\mathcal{F}_n\}_{n\geqslant 0}$. Denote its martingale difference by $\Delta_n f = f_n - f_{n-1}$ ($n \geqslant 0$, with convention $f_{-1} \equiv 0$ and $\mathcal{F}_{-1} = \{\varnothing, \Omega\}$). Its maximal function and conditional quadratic variation are defined by

$$M_n(f) := \sup_{0\leqslant k\leqslant n} |f_k|, \quad M(f) := \sup_{k\geqslant 0}|f_k|,$$

$$s_n(f) := \left(\sum_{k=0}^n E(|\Delta_k f|^2|\mathcal{F}_{k-1})\right)^{1/2}, \quad s(f) := \left(\sum_{k=0}^\infty E(|\Delta_k f|^2|\mathcal{F}_{k-1})\right)^{1/2}.$$

For any $0 < p < \infty$ and $0 < q \leqslant \infty$, the Hardy–Lorentz space of martingales $H^s_{p,q}$ is defined by

$$H^s_{p,q} := \{f = \{f_n\}_{n\geqslant 0} : \|f\|_{H^s_{p,q}} := \|s(f)\|_{p,q} < \infty\}.$$

Definition 2.1. Define the following classes of processes $v = \{v_n\}_{n\geqslant -1}$ adapted to $\{\mathcal{F}_n\}_{n\geqslant -1}$ by

$$V_{p,q} = \{v : \|v\|_{V_{p,q}} = \|M(v)\|_{p,q} < \infty\}, \quad 0 < p < \infty, \quad 0 < q \leqslant \infty,$$

where $M(v) := \sup_{n\geqslant -1}|v_n|$. The martingale transform operator T_v for given martingale f and $v \in V_{p,q}$ is defined by $T_v(f) := \{T_v(f_n)\}_{n\geqslant 0}$, where

$$T_v(f_n) := \sum_{k=0}^n v_{k-1}\Delta_k f, \quad n \geqslant 0.$$

Lemma 2.1 (Grafakos, 2003)**.** *Let* $0 < p < \infty$ *and* $0 < q < r \leq \infty$. *Then there exists constant* $c_{p,q,r}$ *(that depends on* p, q, r*) such that*

$$\|f\|_{p,r} \leqslant c_{p,q,r} \cdot \|f\|_{p,q}.$$

Lemma 2.2 (Grafakos, 2003)**.** *For all* $0 < p, r < \infty$ *and* $0 < q \leqslant \infty$, *we have*

$$\||f|^r\|_{p,q} = \|f\|^r_{pr,qr}.$$

Lemma 2.3 (Grafakos, 2003 *Hölder's Inequality*)**.** *For all* $f \in L_{p_2,q_2}$, $g \in L_{p,q}$, $0 < p, p_2 < \infty$, $0 < q, q_2 \leq \infty$, *with* $\frac{1}{p_1} = \frac{1}{p_2} + \frac{1}{p}$, $\frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q}$, *we have*

$$\|fg\|_{p_1,q_1} \leqslant C_{p_2,q_2,p,q} \cdot \|f\|_{p_2,q_2} \cdot \|g\|_{p,q}.$$