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Convergence in total variation distance for (in)homogeneous Markov processes

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ABSTRACT

In this paper, we study the rate of convergence in total variation distance for time continuous Markov processes, by using some I_ψ and $I_{\psi,t}$-inequalities. For homogeneous reversible process, we use some homogeneous inequalities, including the Poincaré and relative entropy inequalities. For the time-inhomogeneous diffusion process, we use some inhomogeneous inequalities, including the time-dependent Poincaré and Log-Sobolev inequalities. This extends some results for the time-homogeneous diffusion processes in Cattiaux and Guillin (2009).

1. Introduction

In the recent work (Cattiaux and Guillin, 2009), Cattiaux and Guillin obtained, for homogeneous continuous Markov processes, the rate of convergence to the invariant measure in total variation distance, by using functional inequalities. The authors considered the ergodic *diffusion processes*, which means that the infinitesimal generator L satisfies the hypotheses: for $\Psi \in C^2$ and smooth f,

$$L\Psi(f) = \frac{\partial \Psi}{\partial x}(f)Lf + \frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2}(f)\Gamma(f), \quad \text{and} \quad \Gamma(\Psi(f)) = (\Psi'(f))^2\Gamma(f),$$

where Γ is the corresponding *carré du champ* operator associated to L, see Definition 2.2.

In the present paper, we extend their results in two directions. Firstly, we consider the general but *reversible* Markov processes. Secondly, we consider the *inhomogeneous* diffusion process. The inhomogeneity brings more flexibility and complex behaviors might appear consequently, see Saloff-Coste and Zúñiga (2007, 2009, 2011) for inhomogeneous finite Markov chains, and Arnaudon et al. (2008), Cheng and Zhang (2016) for inhomogeneous diffusion. In both cases, we give the

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rates of convergence in the total variation distance by using corresponding inhomogeneous functional inequalities. Indeed, to get the convergence rates, we will also use two elementary lemma, Lemmas 3.2 and 3.3 for the time homogeneous and time-inhomogeneous case, respectively. There is no doubt that we follow closely the idea of Cattiaux and Guillin (2009). However, in the inhomogeneous case, we keep the same assumptions as in Cattiaux and Guillin (2009). We deal with the diffusion processes, without the reversible assumption. Since the generators are time-dependent, the reversible probability measures should be time-dependent, which is not helpful for the convergence. See Saloff-Coste and Zúñiga (2007) for example of inhomogeneous random walks, which admit the reversible probability measures time by time. We use the reversibility instead of the property of diffusion, while in the time-inhomogeneous case what we get is new. Finally, let us mention (Collet and Malrieu, 2008), where a commutation relation in the time-inhomogeneous case was given to guarantee the time-dependent Φ-Sobolev inequalities.

The paper is organized as follows. In Section 2, we present the main results of our paper on the time-homogeneous reversible Markov process and time-inhomogeneous diffusion process. In Section 3, we give the proofs of main theorems.

2. Main results

First, let us introduce some basic notations and definitions, used through this paper. We use C or C' for a positive constant, which may vary from line to line, and will indicate (in the subscript) the dependent parameters. We denote the entropy and oscillation of a (probability density) function f, respectively by

$$\mathrm{Ent}_\mu(f) = \int f \log f \, d\mu \quad \text{and} \quad \mathrm{Osc}(f) = \sup_{x,y} |f(x) - f(y)| \,.$$

We always let ψ be a fixed convex function in $C^2(\mathbb{R}^+)$, which satisfies

- ψ is locally uniform convex, i.e. for any $A > 0$, $\inf_{[0,A]} \psi'' > 0$.
- $\psi(1) = 0$ and $\lim_{x\to\infty} \psi(x)/x = \infty$.

Definition 2.1. Given a measurable space $(E, \mathcal{E})$, let $\mathscr{P}(E)$ be the total of probability measures. For any $\mu_1, \mu_2 \in \mathscr{P}(E)$, we define the total variation distance between μ_1 and μ_2 by

$$\|\mu_1 - \mu_2\|_{TV} = \sup_{A\in\mathcal{E}} \{ |\mu_1(A) - \mu_2(A)| \}.$$

Our results below differ from that in Cattiaux and Guillin (2009) in two folds. Firstly, in Section 2.1, we give the convergence rate in the total variation distance for time-homogeneous *reversible* Markov processes, in contract the diffusion processes in Cattiaux and Guillin (2009). Secondly, in Section 2.2, we give the merging in the total variation distance for *time-inhomogeneous* diffusion processes. Here the merging means the convergence from different initial laws, and we refer to Saloff-Coste and Zúñiga (2007, 2009, 2011) for the case of finite Markov chains

2.1. Time-homogeneous reversible Markov processes

Assume the time-homogeneous Markov process $(X_t, \mathbb{P}_x)$ is reversible with respect to its stationary distribution μ, for more details, see Wang (2004). Let $(P_t)_{t\geq 0}$ be the associated semi-group with the infinitesimal generator $(L, D(L))$ in $L^2(\pi)$, that is

$$\int_E f P_t g d\mu = \int_E g P_t f d\mu, \quad f, g \in L^2(\mu) \quad \text{and} \quad \int_E f L g d\mu = \int_E g L f d\mu, \quad f, g \in D(L).$$

For any probabilistic density h w.r.t. μ, the law of X_t with initial distribution $hd\mu$ is given by $P_t h d\mu$. We also use notations $I_\psi(t, h) = \int \psi(P_t h) d\mu$ and $I_\psi(h) = I_\psi(0, h) = \int \psi(h) d\mu$. The typical examples of reversible Markov processes include the ergodic reversible diffusion processes in Wang (2004), reversible Markov jump processes in Chen (2004). We also refer to these references for the functional inequalities.

Definition 2.2. We define a bilinear form

$$\Gamma(f, g) = \frac{1}{2}(L(fg) - gLf - fLg), \quad f, g \in D(L),$$

the homogeneous carré du champ operator associated to the semigroup or the infinitesimal generator. We use abbreviation $\Gamma(f) = \Gamma(f, f)$.

The carré du champ operator plays a crucial role in the analysis and geometry of Markov processes. The simplest example is that the Euclidean Laplacian Δ gives rise to the standard carré du champ operator $\Gamma(f, g) = \nabla f \cdot \nabla g$, the usual scalar product of the gradients. For further information, we refer to the original paper (Bakry and Émery, 1984) and a recent monograph (Bakry et al., 2014).

Now, we can state our first main theorem for the time-homogeneous reversible Markov processes.