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Reaching goals under ambiguity: Continuous-time optimal portfolio selection

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ABSTRACT

In this paper, the problem of reaching goals for an investor in the financial market is studied. We follow the context of Jin and Zhou (2015) who studied the continuous-time optimal portfolio selection in which the appreciation rates are only known to be in a certain convex closed set. The cost functional in our problem is of type $I_{\{x\geq 1\}}$ which is not concave or convex, even not continuous, we prove that the min–max theorem is still applicable. Via PDE approach, we construct the optimal portfolio strategy. At last, we obtain the saddle point for our problem explicitly.

1. Introduction

Among many investment objectives, institutional money managements are interested in investing the financial market in order to achieve a specific goal practically. The problem of maximizing the probability of reaching a wealth level by a fixed terminal time has been extensively studied by many authors (see Browne, 1999; Karatzas, 1997; Kulldorff, 1993; Spivak and Cvitanic, 1999 and the references therein). In these works, all the parameters of the financial market are given endogenously. In reality, the parameter values are estimated from the historical asset prices. There are several ways to estimate the volatility of the stocks, but it is difficult to estimate the appreciation rates. So one can hardly know the actual values of the appreciation rates. In order to incorporate some degree of ambiguity about the stock-appreciation-rates, we assume that the appreciation rates are only known to be in a convex closed set as in Jin and Zhou (2015). Alternatively, Bouchard et al. (2009) study the problem of minimizing the initial cost of a controlled process which guarantees to reach a target with a given probability. Readers who are interested in stochastic target problems can also consult the seminal book (Touzi, 2012).

In this paper, we consider the problem of reaching goals for an investor under ambiguity. The investor seeks a robust portfolio strategy to maximize the probability of a specified goal under the worst scenario. That is to say, the cost functional in our problem is of type $I_{\{x\geq 1\}}$ which is not concave or convex, even not continuous, we prove that the min–max theorem is still applicable for our problem. Furthermore, we also construct the optimal portfolio strategy and obtain the saddle point for our problem explicitly.

Jin and Zhou (2015) studied the continuous-time optimal portfolio selection problem under ambiguity. When the utility function u is continuous differentiable increasing, strictly concave and satisfies the Inada condition, they formulated the optimal portfolio selection as a min–max problem and obtained the explicit solutions. Especially, they constructed neatly and ingeniously the optimal portfolio strategy via PDE approach.

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The paper is organized as follows. In Section 2, we introduce the formulation of our problem. We give our main results in Section 3.

2. Problem formulation

Let $W(\cdot) = (W_1(\cdot), \dots, W_d(\cdot))'$ be a standard d-dimensional Brownian motion defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. We assume $\mathcal{F}_t = \sigma\{W(s) : 0 \leq s \leq t\} \bigvee \mathcal{N}(P)$ where $\mathcal{N}(P)$ is the collection of P-null sets and set $\mathbf{F} = \{\mathcal{F}_t; 0 \leq t \leq T\}$.

Consider a financial market where $d+1$ assets are traded continuously. One of the assets is a bond, whose price process $S_0(t)$ is assumed for simplicity to be equal to one. The other d assets are non-dividend-paying stocks, whose prices are stochastic processes $S_i(t),\ i = 1, \dots, d$ governed by the following stochastic differential equation (SDE for short):

$$dS_i(t) = S_i(t)\Big(\bar{\mu}_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)\Big),\ t \in [0, T];\ S_i(0) = s_i > 0, \tag{2.1}$$

where $\mu = \{\bar{\mu}(t)' = (\bar{\mu}_1(t), \dots, \bar{\mu}_d(t)),\ t \in [0, T]\}$ is the appreciation rate of the stocks which is $\mathcal{F}_t$ adapted, bounded, and the $d \times d$ matrix $\sigma(t) = (\sigma_{ij}(t))_{1\leq i,j\leq d}$ is the disperse rate of the stocks. Here $'$ denotes the transpose operator. Define

$$\mathbf{G} = \{\mathcal{F}^S_t; 0 \leq t \leq T\},\ \text{where } \mathcal{F}^S_t = \sigma(S(s) : 0 \leq s \leq t) \bigvee \mathcal{N}(P).$$

As explained in Jin and Zhou (2015), the notorious mean-blur problem in estimating the appreciate rate makes it impossible to estimate $\bar{\mu}(\cdot)$ to within a workable accuracy by using pure statistical method based on historical data of the stock prices. So the appreciation rate $\bar{\mu}(t)' := (\bar{\mu}_1(t), \dots, \bar{\mu}_d(t))$ is random and not observable by the investor. Similar to Jin and Zhou (2015), we model the ambiguity about the appreciation rate by assuming that at any time, only a range of the appreciation rate can be estimated, in which the true appreciation rates reside.

Let $\mathcal{B}(\cdot)$ be a given deterministic, measurable and closed-convex-set-valued function on $[0, T]$, and $\mathcal{B}(t)$ is bounded uniformly in t. Define Θ to be the ambiguity set of all possible risk premium processes:

$$\theta(\cdot) := \sigma(\cdot)^{-1}\mu(\cdot) : \mu(\cdot) \text{ is } \mathbf{F}\text{-progressively measurable},\ \mu(t) \in \mathcal{B}(t),\ 0 \leq t \leq T.$$

The investors can observe only the price processes of all the assets traded in the market. Therefore, an admissible portfolio should be $\mathcal{F}^S_t$-adapted. Notice that in general, $\mathcal{F}_t$ is strictly larger than $\mathcal{F}^S_t$. We assume that an investor can decide at time $t \in [0, T]$ what amount $\pi_i(t)$ of his wealth to invest in the ith stock, $i = 1, \dots, d$. And the process $\pi(\cdot) = (\pi_1(\cdot), \dots, \pi_d(\cdot))'$ is $\mathbf{G}$-progressively measurable. The wealth process $X(\cdot) \equiv X^{x,\pi}(\cdot)$ of the investor who is endowed with initial wealth $x \in [0, 1]$ satisfies the linear stochastic differential equation:

$$\begin{aligned} dX(t) &= \sum_{i=1}^{d} \pi_i(t)\frac{dS_i(t)}{S_i(t)} \\ &= \pi(t)'\bar{\mu}(t)dt + \pi(t)'\sigma(t)dW(t). \end{aligned} \tag{2.2}$$

We make the following assumptions throughout this paper:

Assumption 2.1. (i) $\sigma(\cdot)$ is deterministic, and there exists a constant $\delta > 0$ such that $\sigma(t)\sigma(t)' > \delta I_d$, $\forall t \in [0, T]$.
(ii) The true appreciate rate $\bar{\mu}(t)$ is $\mathcal{F}_t$-adapted and $\bar{\mu}(t) \in \mathcal{B}(t),\ \forall t \in [0, T],\ a.s.$
(iii) $0 \notin \mathcal{B}(t),\ 0 \leq t \leq T$.

Remark 2.2. In a financial market, the appreciate rate μ of stocks is usually bigger than the risk free rate r ($r = 0$ in this paper), and only in extreme cases $\mu \leq r$. So Assumption 2.1(iii) holds in most financial markets. And we do not know whether the results in this paper hold or not without Assumption 2.1(iii), especially it is hard to obtain π^* in (3.9).

In the classical problems of reaching goals, the true risk premium $\bar{\theta}$ is known and the investor chooses an optimal portfolio to maximize the probability $P(X(T) \geq 1)$ in which 1 is the prespecified level of the terminal wealth X (refer to Browne, 1999; Karatzas, 1997; Kulldorff, 1993; Spivak and Cvitanic, 1999). In this paper, we assume that the investor cannot observe $\bar{\theta}$ and only knows a range of the risk premium. Thus, in this ambiguous environment, we formulate the problem of reaching goal as follows:

Problem (A).

$$\sup_{\pi\in\mathcal{A}(x)} \inf_{\theta\in\Theta} P(X^{x,\pi}(T)\big|_{\bar{\theta}=\theta} \geq 1), \tag{2.3}$$

where $\mathcal{A}(x)$ is the class of the admissible portfolio processes

$$\pi : [0, T] \times \Omega \to \mathbb{R}^d : \pi(t) \text{ is } \mathcal{F}^S_t\text{-adapted}, \int_0^T |\sigma(t)'\pi(t)|^2 dt < \infty, \text{and } X(t) \geq 0,\ 0 \leq t \leq T,\ a.s. \tag{2.4}$$