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# Lower bounds for the rate of convergence for continuous-time inhomogeneous Markov chains with a finite state space

A.I. Zeifman [a,b,c,*], V.Yu. Korolev [d,b,e], Ya.A. Satin [a], K.M. Kiseleva [f]

[a] *Vologda State University, Russian Federation*
[b] *Institute of Informatics Problems, Federal Research Center "Informatics and Control", Russian Academy of Sciences, Russian Federation*
[c] *Vologda Research Center, Russian Academy of Sciences, Russian Federation*
[d] *Faculty of Computational Mathematics and Cybernetics, Lomonosov Moscow State University, Russian Federation*
[e] *Hangzhou Dianzi University, China*
[f] *Peoples Friendship University of Russia (RUDN University), Vologda State University, Russian Federation*

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ABSTRACT

An approach is proposed to the construction of general lower bounds for the rate of convergence of probability characteristics of continuous-time inhomogeneous Markov chains with a finite state space in terms of special "weighted" norms related to total variation. We study the sharpness of these bounds for finite birth–death–catastrophes process and for a Markov chain with large output intensity from a state.



## 1. Introduction

As is known, the problem of construction of sharp bounds for the rate of convergence of characteristics of Markov chains to their limiting versions is very important. First of all, usually, it is easier to calculate the limit characteristics of a process than to find the exact distribution of state probabilities. Therefore, it is very important to have a possibility to use the limit characteristics as asymptotic approximations for the exact distribution. So, one should have sharp bounds for the rate of convergence in order to determine the time beginning with which this approximation becomes reasonable and accurate enough.

There are many papers dealing with upper bounds for the rate of convergence, see, for instance, Van Doorn et al. (2010), Kartashov (1985) and Mitrophanov (2003, 2004). Moreover, there are a number of different approaches to obtaining upper bounds for the convergence time of Markov chains in different contexts, including the "cut-off phenomenon", Markov chain Monte Carlo algorithms, etc., see, for instance Diaconis and Saloff-Coste (1996, 2006), Van Doorn (2015) and the references therein. Some lower bounds for mixing times in the homogeneous discrete-time case have been discussed in Levin and Peres (2017).

* Correspondenece to: Lenina str, 15, Vologda State University, Vologda, 160000, Russian Federation.
*E-mail address:* a_zeifman@mail.ru (A.I. Zeifman).

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Usually, either the limiting mode is determined (Chakravarthy, 2017), or, on the contrary, the state probabilities are determined as functions of time $t$ and initial conditions (Di Crescenzo et al., 2016).

Let the process be homogeneous and ergodic, and let all the eigenvalues of the intensity matrix, say $s_k$, $k = 0, \ldots, N$, be different, where $s_0 = 0$ and $Res_N \le Res_{N-1} \le Res_1 < 0$. For instance, as it is well known, all the eigenvalues of the intensity matrix for a birth–death process are real and different. Then any solution of the corresponding forward Kolmogorov system can be written as $\sum_k e^{s_k t}\mathbf{c}_k$, where $\mathbf{c}_k$ are the corresponding eigenvectors. Hence, the maximum and minimum rate of convergence are bounded by $C_1 e^{s_N t}$ and $C_1 e^{s_1 t}$ respectively. However, even in this case the construction of estimates for the spectrum is a rather difficult problem. In Zeifman and Korolev (2015) we presented an approach for finding sharp upper bounds in natural metrics via essential positivity of reduced intensity matrix of a Markov chain. These bounds are sharp for nonnegative difference of initial conditions of two probability distributions of Markov chain. However, in a general situation the assumption of the nonnegativity of this difference does not hold and the real lower bound of the rate of convergence can be essentially smaller. At the same time, lower bounds for the convergence rate are very important, because they give an opportunity to determine the time until which the accurate approximation of transient probability characteristics by the limiting probability characteristics is impossible.

As far as we know, these bounds were studied previously only in the papers (Zeifman, 1995a; Granovsky and Zeifman, 2000, 2004, 2005; Zeifman et al., 2013) for birth–death processes via the notion of logarithmic norm of an operator and the corresponding estimated. In this paper we deal with the weak ergodicity of (as a rule, inhomogeneous) continuous-time Markov chains with a general structure of the infinitesimal matrix and formulate an algorithm for the construction of lower bounds for the rate of convergence in the special weighted norms related to total variation.

Our general approach is closely connected with the notion of the logarithmic norm and the corresponding bounds for the Cauchy matrix (Zeifman, 1985; 1995a; 1995b; Granovsky and Zeifman, 1997, 2004; Zeifman et al., 2006, 2013, 2014), and allows to obtain general upper and lower bounds for the rate of convergence for a countable state space and general initial conditions. On the other hand, if the state space of the chain is finite, then the desired bounds can be obtained in a simple way without this special technique.

Section 2 contains preliminary material on the logarithmic norm and the corresponding lower bound on the rate of convergence for a linear differential system. The general result is presented in Section 3. In Section 4 some important classes of continuous-time Markov chains are considered for which it is possible to obtain exact lower convergence rate estimates.

## 2. Logarithmic norm and related bounds

The concept of the logarithmic norm of a square matrix was developed independently by Dahlquist (1958) and Lozinskiĭ (1958) as a tool to derive error bounds in the numerical integration of initial-value problems for a system of ordinary differential equations (see also the survey papers Ström, 1975 and Söderlind, 2006). The logarithmic norm for an operator function and the corresponding bounds for solutions of differential equation in Banach space has been developed by Dalecki and Krein (1974). One can find detailed discussion in Van Doorn et al. (2010), Granovsky and Zeifman (2004) and Zeifman (1995b).

Let $G(t)$ be a bounded for almost all $t \ge 0$ operator function on a Banach space $\mathcal{B}$ endowed with the corresponding norm $\|\cdot\|$. Then

$$\gamma(G(t))_{\mathcal{B}} = \lim_{h\to+0} \frac{\|I + hG(t)\| - 1}{h} \tag{1}$$

is called the logarithmic norm of $G(t)$.

On the other hand, if we consider the Cauchy operator $V(t, s)$ of the differential equation $\frac{d\mathbf{x}}{dt} = G(t)\mathbf{x}$, i.e., $\mathbf{x}(t) = V(t, s)\mathbf{x}(s)$ for any $s, t$, then

$$\gamma(G(t))_{\mathcal{B}} = \lim_{h\to+0} \frac{\| V(t+h, t) \| -1}{h}, \quad t \ge 0. \tag{2}$$

One can prove (see the above references) that both limits (1) and (2) exist and coincide. Moreover, if $\mathcal{B} = l_1$, then $G(t) = (g_{ij}(t))_{i,j=0}^{\infty}$ and

$$\gamma(G(t)) = \sup_j (g_{jj}(t) + \sum_{i\neq j} | g_{ij}(t) |).$$

The corresponding inequality $\|V(t,s)\| \le \exp\left\{\int_s^t \gamma(G(\tau))\, d\tau\right\}$, was used in our previous papers for the construction of upper bounds for the rate of convergence of continuous-time Markov chains.

Here we are interested in a lower bound. We have

$$\begin{aligned}
\|V(\tau+h,s)\| - \|V(\tau,s)\| &= \|V(\tau+h,s)\| - \left\|V^{-1}(\tau+h,\tau)\,V(\tau+h,s)\right\| \\
&\ge \|V(\tau+h,s)\| - \left\|V^{-1}(\tau+h,\tau)\right\| \|V(\tau+h,s)\| \\
&= \|V(\tau+h,s)\| \left(1 - \left\|V^{-1}(\tau+h,\tau)\right\|\right).
\end{aligned}$$