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On relative log-concavity and stochastic comparisons[☆]

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ABSTRACT

In this paper, we investigate the equivalence among stochastic orderings between random variables under the scenario where the ratio of density functions or survival functions is log-concave. Some sufficient and necessary conditions that may be easily checked for the usual stochastic order, the hazard rate order, the mean residual lifetime order, the harmonic mean residual lifetime order and the mean inactivity time order are presented. As for illustration, we also provide several numerical examples and three applications.

1. Introduction

The behaviour of the likelihood ratio of two distributions has interest in several researching areas, such as reliability, stochastic processes, risk theory, statistical inference, and so on. See Ross (1996) and Shaked and Shanthikumar (2007) for reference. Typically, in the literature there are two lines of studies on different behaviour of the likelihood ratio and its applications. The first one focuses on the log-concavity of the likelihood ratio, which was firstly introduced and named as *the relative log-concavity order* in Whitt (1985) to study the variability orderings between random variables. Following the same line, Yu (2009) employed the relative log-concavity order to identify sufficient and necessary conditions regarding the existence of the likelihood ratio order, the reversed hazard rate order, the hazard rate order and the usual stochastic order, and equivalence among them. Later, Yu (2010) further investigated properties of the relative log-concavity order and derived a pair of triangle inequalities regarding the Kullback–Leibler divergence. Research along the other line on the study of likelihood ratio's behaviour focuses on the unimodality instead of the log-concavity. Metzger and Rüshendorf (1991) showed that if the likelihood ratio is unimodal, then the ratio of the corresponding survival functions is also unimodal. Recently, Belzunce and Martínez-Riquelme (2016) proved that the finding in Metzger and Rüshendorf (1991) is not true in general and the ratio of survival functions could be monotonic. They also provided several sufficient conditions for the scenario where the hazard rate order exists while the likelihood ratio order does not hold, and where the mean residual lifetime order exists but the hazard rate order does not hold.

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In the literature, stochastic orders and their relationship have been studied extensively. However, it is usually not easy and sometimes even surprisingly difficult to verify the existence of stochastic ordering for specific distributions, especially for those without a closed-form distribution function or survival function. By employing the log-concavity of likelihood ratio, Yu (2009) derived several sufficient and necessary conditions regarding the existence of several stochastic orders and succeeded in applying these characterizations to stochastically compare the exponential family distributions, the convolutions of gamma distributions, the convolutions of negative binomial distributions and the corresponding mixtures, respectively. The findings of Yu (2009) reveal the availability and usefulness of the relative log-concavity for studying stochastic ordering. Motivated by Yu (2009), in this paper we further study the characterization and equivalence among stochastic orders. Specifically, our study is two-fold. On one hand, partially following the line on log-concave likelihood ratio and along with Yu (2009), we further derive equivalence among several stochastic orders and provide an application of the relative log-concavity order in reliability theory. On the other hand, in view of the fact that likelihood ratio may not always be log-concave, we propose an alternative approach and focus on situations with log-concave survival functions' ratio. Based on the presented approach, we derive some sufficient and necessary conditions comparing random variables with respect to various stochastic orders, and we also provide two applications of these theoretical results. One is on the stochastic comparison between exponential distribution and its mixture, and the other is on comparing sample spacings from a heterogeneous exponential sample and a homogeneous one.

The rest of the paper is as follows. In Section 2 we recall several basic concepts and some useful lemmas. Section 3 contains the main results on equivalence among several stochastic orders, including the likelihood ratio order, the hazard rate order, the reversed hazard rate order, the usual stochastic order, the (harmonic) mean residual lifetime order and the mean inactivity time order. Finally, three applications illustrating the effectiveness of the theoretical results are presented in Section 4.

Throughout this paper, all random variables are assumed to be nonnegative, the inverse functions are assumed to be right continuous and the terms 'increasing' and 'decreasing' mean 'nondecreasing' and 'nonincreasing', respectively.

2. Preliminaries

In this section we introduce some basic notations and concepts to be used in the sequel.

Definition 2.1. Consider two absolutely continuous random variables X and Y with probability density functions (PDFs) or discrete random variables with probability mass functions (PMFs) f and g, respectively. Denote their respective distribution functions as F and G, survival functions by $\bar{F}$ and $\bar{G}$, and expectations by μ_X and μ_Y. Then, X is said to be smaller than Y in the

(i) *usual stochastic order*, denoted as $X \le_{\rm st} Y$, if $\bar{F}(t) \le \bar{G}(t)$ for all t;
(ii) *hazard rate order*, denoted as $X \le_{\rm hr} Y$, if $f(x)/\bar{F}(t) \ge g(x)/\bar{G}(t)$ for all t;
(iii) *reversed hazard rate order*, denoted as $X \le_{\rm rh} Y$, if $f(x)/F(t) \le g(x)/G(t)$ for all t;
(iv) *likelihood ratio order*, denoted as $X \le_{\rm lr} Y$, if $f(t)/g(t)$ is decreasing in t;
(v) *mean residual lifetime order*, denoted as $X \le_{\rm mrl} Y$, if $\frac{\int_t^{+\infty} \bar{F}(x)\,{\rm d}x}{\bar{F}(t)} \le \frac{\int_t^{+\infty} \bar{G}(x)\,{\rm d}x}{\bar{G}(t)}$ for all t;
(vi) *harmonic mean residual lifetime order*, denoted as $X \le_{\rm hmrl} Y$, if $\frac{\int_t^{+\infty} \bar{F}(x)\,{\rm d}x}{\mu_X} \le \frac{\int_t^{+\infty} \bar{G}(x)\,{\rm d}x}{\mu_Y}$ for all t;
(vii) *mean inactivity time order*, denoted by $X \le_{\rm mit} Y$, if $\frac{\int_0^t F(x)\,{\rm d}x}{F(t)} \ge \frac{\int_0^t G(x)\,{\rm d}x}{G(t)}$ for all t.

For book-length treatments of properties and applications of stochastic orders, please refer to Shaked and Shanthikumar (2007) and Li and Li (2013).

Definition 2.2. Consider two absolutely continuous (discrete) random variables X and Y defined on $\mathbb{R}$ ($\mathbb{Z}$) with PDFs (PMFs) f and g, respectively. Suppose the support of X is a subset of that of Y, then X is said to be log-concave relative to Y (denoted as $X \le_{\rm lc} Y$), if $\log(f(x)/g(x))$ is a concave function on the support of X.

Lemma 2.3 (*Theorems 1 and 2 in Yu, 2009*). *Assume two nonnegative absolutely continuous (discrete) random variables X and Y have PDFs (PMFs) f and g, respectively. If $X \le_{\rm lc} Y$ and $\ell_0(x) = \log f(x)/g(x)$ is continuously differentiable, then*

(i) $X \le_{\rm rh} Y \Longleftrightarrow X \le_{\rm lr} Y \Longleftrightarrow \lim_{x\to 0^+} \ell_0'(x) \le 0 \left(\frac{f(1)}{g(1)} \le \frac{f(0)}{g(0)}\right)$;
(ii) $X \le_{\rm st} Y \Longleftrightarrow X \le_{\rm hr} Y \Longleftrightarrow \lim_{x\to 0^+} \ell_0(x) \ge 0 \left(\frac{f(0)}{g(0)} \ge 1\right)$.

In addition to the characterization results of stochastic orders quoted in Lemma 2.3, Yu (2009) also applied the relative log-concavity order to stochastic comparisons between exponential family distributions, convolution of gamma distributions, convolutions of negative binomial distributions and their mixtures, respectively. As demonstrated by these applications, the relative log-concavity order can provide an effective way to establish ordering relations. In the context of $X \le_{\rm lc} Y$, the following theorem develops more characterizations of stochastic orderings between random variables and serves as a