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On the existence of optimal controls for backward stochastic partial differential equations

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ABSTRACT

This paper is concerned with the existence of optimal controls for backward stochastic partial differential equations with random coefficients, in which the control systems are represented in an abstract evolution form, i.e. backward stochastic evolution equations. Under some growth and monotonicity conditions on the coefficients and suitable assumptions on the Hamiltonian, the existence of the optimal control boils down to proving the uniqueness and existence of a solution to the stochastic Hamiltonian system, i.e. a fully coupled forward–backward stochastic evolution equation. Using some a prior estimates, we prove the uniqueness and existence of the solution via the method of continuation. Two examples of linear–quadratic control are solved to demonstrate our results.

1. Introduction

In this paper, we consider an optimal control problem under a stochastic backward system of infinite dimensions. To be more precise, the control system under consideration is given by a backward stochastic partial differential equation in the abstract evolution form:

$$\begin{cases} dy(t) = [A(t)y(t) + B(t)z(t) + D(t)u(t) + G(t)]dt + z(t)dW(t), \\ y(T) = \xi, \end{cases} \tag{1.1}$$

with the cost functional:

$$J(u(\cdot)) = \mathrm{E}\left[\int_0^T l(t, y(t), z(t), u(t))dt + h(y(0))\right], \tag{1.2}$$

where A is a given stochastic evolution operator, B, D, G, ξ, l and h are given random maps, and $W(\cdot)$ is a one-dimensional standard Brownian motion. The state process $(y(\cdot), z(\cdot))$ and the control process $u(\cdot)$ take values in Hilbert spaces $V \times H$ and U, respectively. The objective of the optimal control problem is to find a control process that minimizes the cost functional (1.2) over the set of admissible controls. The work in Meng and Shi (2013) established necessary and sufficient maximum principles for a more general backward control system of infinite dimensions. However, the existence of optimal controls was not discussed thoroughly in Meng and Shi (2013). This paper attempts to fill in this gap and establish the existence condition for an optimal control under the system (1.1)–(1.2).

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The existence of optimal controls is a fundamental problem in stochastic optimal control theory which has attracted comprehensive attention in the past years. One approach to study the existence of optimal controls is based on the dynamic programming principle and the solvability of the corresponding HJB equation in a sufficiently regular sense. The work in El Karoui et al. (1987) used the compactness argument and proved the existence of an optimal Markovian relaxed control for systems with degenerate diffusions. Based on an approximation of stochastic control systems with smooth coefficients, the existence of optimal controls for stochastic control systems was investigated in Buckdahn et al. (2010) with the cost functional given by a controlled backward stochastic differential equation. Also some earlier works along this research line can be found in Bismut (1976) and Davis (1973), and the references therein. However, since all the coefficients in (1.1)–(1.2) are random, the corresponding HJB equation becomes a nonlinear backward stochastic partial differential equation, the solvability of which is still an open problem. Therefore, it is not suitable to follow the dynamic programming principle approach to investigate the underlying problem in our paper.

Another approach relies on the stochastic maximum principle, where the existence of optimal controls is studied by utilizing the stochastic Hamiltonian system. Indeed, the stochastic Hamiltonian system is a fully-coupled forward–backward stochastic differential equation (FBSDE), consisting of a state equation, an adjoint equation and some optimality conditions for the optimal control. Even in the finite-dimensional case, the uniqueness and existence of solutions to nonlinear fully-coupled forward–backward systems is a very challenging problem. There has been many works on this topic, see for example, Hu and Peng (1995), Ma et al. (1994), Pardoux and Tang (1999), Peng and Wu (1999) and the references therein. However, very limited works have focused on the solvability of infinite-dimensional FBSDEs. Guatteri (2007) proved that a class of fully-coupled, infinite-dimensional FBSDEs has a local unique solution. Guatteri and Masiero (2013) considered a stochastic optimal control problem for a heat equation with boundary noise, boundary controls and deterministic coefficients. In Guatteri and Masiero (2013), under suitable assumptions on the coefficients, the existence condition of optimal controls was presented in strong sense by solving the associated stochastic Hamiltonian system of infinite dimensions; the bridge method and the auxiliary deterministic Riccati equation were applied to obtain the solution.

In this paper, the stochastic Hamiltonian system is described by the following infinite-dimensional FBSDE:

$$\begin{cases} dk(t) = -\left[A^*(t)k(t) + l_y(t, y(t), z(t), u(t))\right]dt \\ \qquad\qquad -\left[B^*(t)k(t) + l_z(t, y(t), z(t), u(t))\right]dW(t), \\ dy(t) = \left[A(t)y(t) + B(t)z(t) + D(t)\gamma(D^*(t)k(t)) + G(t)\right]dt + z(t)dW(t), \\ \ k(0) = -h_y(y(0)), \quad y(T) = \xi \end{cases} \tag{1.3}$$

where A^*, B^* and D^* denote the dual operators of A, B and D, respectively, and γ is a function satisfying suitable conditions, to be specified in Assumption **(A.5)**. Unlike the Hamiltonian system in Guatteri and Masiero (2013), since all the coefficients in (1.3) are random and time-varying, the adaptability of the integrand in the stochastic integral may not be satisfied and the solution of this equation cannot be defined in the mild sense. Instead, we will study FBSDE (1.3) in the sense of weak solution (i.e. in the PDE sense). We first show the existence and uniqueness of a solution to FBSDE (1.3) via using continuous dependence theorems for stochastic evolution equations (SEEs) and backward stochastic evolution equations (BSEEs) in Meng and Shi (2013). Then by stochastic maximum principles in Meng and Shi (2013), the existence of an optimal control is immediately obtained. Compared with existing works on infinite-dimensional FBSDEs (see e.g. Guatteri and Masiero (2013)), the approach developed in our paper is more convenient and much simpler.

The rest of this paper is organized as follows. Section 2 introduces some basic notation, formulates the control problem in an infinite-dimensional backward system and recalls stochastic maximum principles established by Meng and Shi (2013). In Section 3, main results in our paper are presented, and two infinite-dimensional linear–quadratic control problems are solved in Section 4. Section 5 concludes the paper with some remarks.

2. Preliminaries and problem formulation

In this section, we first introduce the basic notation to be used throughout this paper. We formulate the control problem under a state equation described by a backward stochastic partial differential equation (BSPDE) in the abstract evolution form, i.e. a BSEE. At the end of this section, we give necessary and sufficient maximum principles for our control system.

First of all, we fix a complete probability space $(\Omega, \mathscr{F}, P)$. Let $W(\cdot) \triangleq \{W(t)\}_{t\geq 0}$ be a one-dimensional standard Brownian motion defined on $(\Omega, \mathscr{F}, P)$. We further equip $(\Omega, \mathscr{F}, P)$ with a filtration $\mathbb{F} \triangleq \{\mathscr{F}_t\}_{t\geq 0}$, which is the natural filtration generated by $W(\cdot)$ and augmented in the usual way. Denote by $\mathscr{P}$ the predictable σ-field on $[0, T] \times \Omega$, $\mathscr{B}(\Lambda)$ the Borel σ-algebra of any topological space Λ, and $\|\cdot\|_H$ the norm of any Hilbert space H. Let T be a finite time horizon, i.e. $0 < T < \infty$. Throughout this paper, we let C and K be two generic constants, which may be different from line to line. We introduce the following spaces on $(\Omega, \mathscr{F}, \mathbb{F}, P)$ for stochastic processes or random variables taking values in any Hilbert space H:

- $M^2_{\mathscr{F}}(0, T; H)$: the set of all $\mathbb{F}$-adapted, H-valued processes $\varphi = \{\varphi(t, \omega),\ (t, \omega) \in [0, T] \times \Omega\}$ such that $\|\varphi\|_{M^2_{\mathscr{F}}(0,T;H)} \triangleq \sqrt{E[\int_0^T \|\varphi(t)\|^2_H dt]} < \infty$;
- $S^2_{\mathscr{F}}(0, T; H)$: the set of all $\mathbb{F}$-adapted, H-valued, càdlàg processes $\varphi = \{\varphi(t, \omega),\ (t, \omega) \in [0, T] \times \Omega\}$ such that $\|\varphi\|_{S^2_{\mathscr{F}}(0,T;H)} \triangleq \sqrt{E[\sup_{0\leq t\leq T} \|\varphi(t)\|^2_H]} < +\infty$;