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New and refined bounds for expected maxima of fractional Brownian motion

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ABSTRACT

For the fractional Brownian motion B^H with the Hurst parameter value H in $(0, 1/2)$, we derive new upper and lower bounds for the difference between the expectations of the maximum of B^H over $[0, 1]$ and the maximum of B^H over the discrete set of values in^{-1}, $i = 1, \dots, n$. We use these results to improve our earlier upper bounds for the expectation of the maximum of B^H over $[0, 1]$ and derive new upper bounds for Pickands' constant.

1. Introduction

Let $B^H = (B_t^H)_{t\ge 0}$ be a fractional Brownian motion (fBm) process with Hurst parameter $H \in (0, 1)$, i.e. a zero-mean continuous Gaussian process with the covariance function $\mathbf{E}B_s^H B_t^H = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H})$, $s, t \ge 0$. Equivalently, the last condition can be stated as $B_0^H = 0$ and

$$\mathbf{E}(B_s^H - B_t^H)^2 = |s-t|^{2H}, \quad s, t \ge 0. \tag{1}$$

Recall that the Hurst parameter H characterizes the type of the dependence of the increments of the fBm. For $H \in (0, \frac{1}{2})$ and $H \in (\frac{1}{2}, 1)$, the increments of B^H are respectively negatively and positively correlated, whereas the process $B^{1/2}$ is the standard Brownian motion which has independent increments. The fBm processes are important construction blocks in various application areas, the ones with $H > \frac{1}{2}$ being of interest as their increments exhibit long-range dependence, while it was shown recently that fBm's with $H < \frac{1}{2}$ can be well fitted to real life telecommunications, financial markets with stochastic volatility and other financial data (see, e.g., Araman and Glynn, 2012; Bayer et al., 2016). For detailed exposition of the theory of fBm processes, we refer the reader to Biagini et al. (2008), Mishura (2008), Nourdin (2012) and references therein.

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Computing the value of the expected maximum

$$M^H := \mathbf{E} \max_{0\le t\le 1} B_t^H$$

is an important question arising in a number of applied problems, such as finding the likely magnitude of the strongest earthquake to occur this century in a given region or the speed of the strongest wind gust a tall building has to withstand during its lifetime etc. For the standard Brownian motion $B^{1/2}$, the exact value of the expected maximum is $\sqrt{\pi/2}$, whereas for all other $H \in (0, 1)$ no closed-form expressions for the expectation are known. In the absence of such results, one standard approach to computing M^H is to evaluate instead its approximation

$$M_n^H := \mathbf{E} \max_{1\le i\le n} B_{i/n}^H, \quad n \ge 1,$$

(which can, for instance, be done using simulations) together with the approximation error

$$\Delta_n^H := M^H - M_n^H.$$

Some bounds for Δ_n^H were recently established in Borovkov et al. (2017). The main result of the present note is an improvement of the following upper bound for Δ_n^H obtained in Theorem 3.1 of Borovkov et al. (2017): for $n \ge 2^{1/H}$,

$$\Delta_n^H \le \frac{2(\ln n)^{1/2}}{n^H}\left(1 + \frac{4}{n^H} + \frac{0.0074}{(\ln n)^{3/2}}\right). \tag{2}$$

Lower bound for Δ_n^H is obtained as well and we study for which H and n upper and lower bounds hold simultaneously. We also obtain a new upper bound for the expected maximum M^H itself and some functions of it, which refines previously known results (see e.g. Borovkov et al., 2017; Shao, 1996), and use it to derive an improved upper bound for the so-called Pickands' constant, which is the basic constant in the extreme value theory of Gaussian processes.

The paper is organized as follows: Section 2 contains the results, with comments and examples, and Section 3 contains the proofs.

2. Main results

From now on, we always assume that $H \in (0, \frac{1}{2})$. The next theorem is the main result of the note. As usual, $\lfloor x\rfloor$ and $\lceil x\rceil$ denote the floor and the ceiling of the real number x.

Theorem 1. *(1) For any $\alpha > 0$ and $n \ge 2^{1/\alpha} \vee (1 + \frac{\alpha}{1+\alpha})^{1/(2\alpha H)}$ one has*

$$\frac{\Delta_n^H}{n^{-H}(\ln n)^{1/2}} \le \frac{(1 - \lfloor n^\alpha\rfloor^{-1})^H (1+\alpha)^{1/2}}{1 - \lfloor n^\alpha\rfloor^{-H}(1 + \alpha/(1+\alpha))^{1/2}}. \tag{3}$$

(2) For any $n \ge 2$ one has

$$\frac{\Delta_n^H}{n^{-H}(\ln n)^{1/2}} \ge n^H\left(\frac{L}{(\ln n^H)^{1/2}} - 1\right)^+, \tag{4}$$

where $L = 1/\sqrt{4\pi e \ln 2} \approx 0.2$ and $a^+ = a \vee 0$.

Remark 1. Note that inequality (4) actually holds for all $H \in (0, 1)$.

Remark 2. Let us study for which H and n upper and lower bounds (3) and (4) hold simultaneously under assumption that (4) is non-trivial. For non-triviality we need to have $n < \exp\frac{L^2}{H}$. In order to have $2^{1/\alpha} \le \exp\frac{L^2}{H}$ we restrict α to $\alpha \ge \frac{H\ln 2}{L^2}$. In order to have $(1 + \frac{\alpha}{1+\alpha})^{1/(2\alpha H)} \le \exp\{\frac{L^2}{H}\}$, or, what is equivalent,

$$\left(1 + \frac{\alpha}{1+\alpha}\right)^{1/\alpha} \le \exp\{2L^2\}, \tag{5}$$

we note that the function $q(\alpha) = (1 + \frac{\alpha}{1+\alpha})^{1/\alpha}$ continuously strictly decreases in $\alpha \in (0, \infty)$ from e to 1, and taking into account the value of L, we get that there is a unique root $\alpha^* \approx 7.48704$ of the equation $\left(1 + \frac{\alpha}{1+\alpha}\right)^{1/\alpha} = \exp\{2L^2\}$ and for $\alpha \ge \alpha^*$ we have that (5) holds. Therefore for $\alpha > \alpha^*$, $H < \frac{\alpha^* L^2}{\ln 2} \approx 0.456$ and $\exp\{\frac{L^2}{H}\} > n > 2^{1/\alpha} \vee (1 + \frac{\alpha}{1+\alpha})^{1/(2\alpha H)}$ we have that lower bound (4) holds and is non-trivial. Moreover, $2^{1/\alpha} < 2^{1/\alpha^*}(\approx 1.097) < \exp\{\frac{L^2}{H}\}$, $(1 + \frac{\alpha}{1+\alpha})^{1/(2\alpha H)} < (1 + \frac{\alpha^*}{1+\alpha^*})^{1/(2\alpha^* H)} = \exp\{\frac{L^2}{H}\}$, so the interval $(2^{1/\alpha} \vee (1 + \frac{\alpha}{1+\alpha})^{1/(2\alpha H)}, \ \exp\{\frac{L^2}{H}\})$ is non-empty and for such n upper bound (3) holds. The only question is if this interval contains the integers. If it is not the case we can increase the value of α. For example, put $H = 0.01, \alpha = 16$, then it holds that the interval $(2^{1/\alpha} \vee (1 + \frac{\alpha}{1+\alpha})^{1/(2\alpha H)}, \ \exp\{\frac{L^2}{H}\}) = (1.044 \vee 7.534, \ 20.085) = (7.534, \ 20.085)$.