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Convergence of series of strongly integrable random variables and applications

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We investigate the convergence of series of random variables with second exponential moments. We give sufficient conditions for the convergence of these series with respect to an exponential Orlicz norm and almost surely. Applying these results to sequences with d-subgaussian increments, we examine the asymptotic behavior of weighted sums of subgaussian random variables in a unified setting.

1. Introduction and main results

Let $\mathcal{X} = \{X_k,\ k \geq 1\}$ be a sequence of real random variables, defined on some probability space $(\Omega, \mathcal{F}, P)$ with exponential moments. We focus on the asymptotic behavior of the random series

$$\sum_{k=1}^{\infty} X_k. \tag{1.1}$$

This problem has been examined in Amini et al. (2004, 2007), Azuma (1967), Chow (1966), Kahane (1985) and Ouy (1976) (resp. in Giuliano Antonini et al., 2008), assuming $\mathcal{X}$ to be a sequence of subgaussian random variables (resp. φ-subgaussian random variables) with some restrictions on the dependence structure of $\mathcal{X}$ such as, independence, m-dependence, negative dependence, and m-acceptability.

In the present work, we study the problem of convergence of the series in (1.1) when $\mathcal{X}$ is a sequence of random variables satisfying some increment condition with respect to an exponential Orlicz norm, defined by the function $\varphi(x) = \exp(x^2) - 1$. When the above series converges, we give some estimates for the Orlicz norms of the limit and the two-sided maximal function of its partial sums. Subgaussian random variables are typical examples of random variables belonging to the exponential Orlicz space $L^{\varphi}(\Omega)$. Our approach provides a unified framework to study the asymptotic behavior of weighted

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series of subgaussian random variables and enables us to improve some results in Amini et al. (2004, 2007), Azuma (1967), Chow (1966), Kahane (1985), Ledoux and Talagrand (2002) and (Taylor and Hu, 1987). We also recover some statements proved in Giuliano Antonini et al. (2008). Finally, we examine an example of a convergent subgaussian series, which is beyond the scope of the last quoted references.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, a Young function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is an increasing convex function with $\psi(0) = 0$ and $\psi(x) \to +\infty$ when $x \to +\infty$. For any Young function ψ we can associate the Orlicz space $L^\psi(\Omega)$, the space of random variable $X : \Omega \longrightarrow \mathbb{R}$ such that $\mathbb{E}\big(\psi(a|X|)\big) < \infty$ for some $0 < a < \infty$, see for instance Buldygin and Kozachenko (2000, Chapter 2). Recall that the space $L^\psi(\Omega)$ is endowed with the norm

$$\forall X \in L^\psi(P), \quad \|X\|_\psi = \inf\Big\{c > 0 \ : \ \mathbb{E}\big(\psi\{\frac{|X|}{c}\}\big) \le 1\Big\}$$

and that $\big(L^\psi(\Omega), \|\cdot\|_\psi\big)$ is a Banach space.

Let E be a non empty set, a pseudo-metric d on E is a positive application on $E \times E$ which has the properties of a distance except that it does not necessarily separate points ($d(s,t) = 0$ does not always imply $s = t$). For a pseudo-metric space (E, d) and $\delta > 0$, the δ-entropy number $N(E, d, \delta)$ is the smallest covering number (possibly infinite) of E by open d-balls of radius δ. R. M. Dudley used this last notion to formulate the following well-known criterion, see e.g. Ledoux and Talagrand (2002), van der Vart and Wellner (2013) and Weber (2004).

Theorem 1.1. *Let E be a countable set, provided with a pseudo-metric d and let $X = \{X_t,\ t \in E\}$ be a stochastic process indexed on E, with a basic probability space $(\Omega, \mathcal{F}, P)$, and satisfying the following increment condition*

$$\forall s, t \in E, \qquad \|X_t - X_s\|_\psi \le d(s,t).$$

Set $diam(E, d) = \sup_{s,t\in E} d(s,t)$ and assume that the entropy integral

$$\mathcal{I}(E, d) = \int_0^{diam(E,d)} \psi^{-1}\big(N(E, d, u)\big)\, du$$

is convergent. Then

$$\Big\| \sup_{s,t\in E} |X_t - X_s| \Big\|_\psi \le 8\mathcal{I}(E, d). \tag{1.2}$$

From now on, φ will denote the Young function defined on $\mathbb{R}_+$ by

$$\varphi(x) = \exp(x^2) - 1.$$

Let $L^\varphi(\Omega)$ be the corresponding Orlicz space and consider a sequence of real random variables $\mathcal{X} = \{X_k,\ k \ge 1\}$ in $L^\varphi(\Omega)$. Set

$$S_n(\mathcal{X}) = \sum_{k=1}^{n} X_k \quad \text{for } n \ge 1 \qquad \text{and} \qquad \mathcal{S}(\mathcal{X}) = \{S_n(\mathcal{X}),\ n \ge 1\}.$$

In this section we are interested in the asymptotic behavior of $\mathcal{S}(\mathcal{X})$ when $\mathcal{X}$ satisfies the following increment condition: *For some sequence $\tilde{u} = \{u_k,\ k \ge 1\}$ of positive real numbers, and some $\alpha > 0$, we have*

$$\forall 0 \le n < m < \infty, \qquad \| \sum_{k=n+1}^{m} X_k \|_\varphi \le \Big(\sum_{k=n+1}^{m} u_k \Big)^\alpha. \tag{1.3}$$

We emphasize that this will be the only condition on $\mathcal{X}$ for the validity of our main results (Theorems 1.2 and 1.3), in particular there is no restriction on the dependence structure of the underlying sequence. We begin by establishing a strong maximal inequality for $\mathcal{S}(\mathcal{X})$.

Theorem 1.2. *Let $\mathcal{X} = \{X_k,\ k \ge 1\}$ be a sequence of random variables satisfying the increment condition (1.3), for some sequence of positive real numbers $\tilde{u} = \{u_k,\ k \ge 1\}$ and some $\alpha > 0$. If the series $\sum_{k\ge1} u_k$ converges, then*

$$\Big\| \sup_{1\le i<j} \Big| \sum_{k=i+1}^{j} X_k \Big| \Big\|_\varphi \le 8C(\alpha) \Big(\sum_{k=1}^{\infty} u_k \Big)^\alpha, \tag{1.4}$$

where $C(\alpha) = \frac{2^{2\alpha+2}}{\sqrt{\alpha}} \int_{\sqrt{\alpha \ln 3}}^{+\infty} x^2 e^{-x^2}\, dx < \infty$.

Proof. Let $\alpha > 0$ and assume $\mathbf{u} := \sum_{k\ge1} u_k < \infty$. Set $S_0 = 0,\ U_0 = 0,\ U_n = \sum_{k=1}^{n} u_k$ for $n \ge 1$ and $\mathcal{U} = \{U_n,\ n \ge 0\}$. We will apply Theorem 1.1 with $E = \mathbb{N}$, endowed with the pseudo metric $d_\varphi(n, m) = \|S_m(\mathcal{X}) - S_n(\mathcal{X})\|_\varphi$ for $n \le m$. To succeed we need to estimate the entropy numbers $\big\{N(\mathbb{N}, d_\varphi, \delta),\ \delta > 0\big\}$. Let $\epsilon > 0$ and denote by d the distance defined on $\mathbb{N}^2$ by $d(i,i) = 0$ for each $i \in \mathbb{N}$ and $d(i,j) = \sum_{k=i\wedge j+1}^{i\vee j} u_k$ otherwise. We will also denote by $B_d(r, \epsilon)$ the open d-ball