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Laplace symbols and invariant distributions

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ABSTRACT

We introduce a new kind of symbol in the framework of Itô processes which are bounded on one side. The connection between this symbol and the infinitesimal generator is analyzed. Based on this concept, an integral criterion for invariant distributions of the underlying process is derived. Some applications are mentioned.

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1. Introduction

Over the last two decades, the so-called symbol of a stochastic process has proven to be a useful tool in order to derive local and global properties of the corresponding stochastic process (cf. Hoh, 1998, Schilling, 1998b, Schnurr, 2013 and Chapter 5 of Böttcher et al., 2013). Following ideas of Jacob (1998) and Schilling (1998a), Schnurr has generalized this concept to homogeneous diffusions with jumps in the sense of Jacod and Shiryaev (2003). In its most general form the symbol of an $\mathbb{R}^d$-valued homogeneous diffusion with jumps $(X_t)_{t\geq 0}$ looks as follows. For $x, \xi \in \mathbb{R}^d$

$$p(x,\xi) := -\lim_{t\downarrow 0} \frac{\mathbb{E}^x e^{i(X_t^\sigma - x)'\xi} - 1}{t}, \tag{1.1}$$

where X^σ is the process X stopped at time σ, the first exit time of a compact neighborhood of x, and x' denotes the transpose of the vector x. Under mild regularity conditions the limit in (1.1) is always a continuous negative definite function in the sense of Schoenberg (cf. Berg and Forst, 1975), which means that for each fixed x we obtain a Lévy–Khintchine exponent

$$\phi(\xi) := -i\ell'\xi + \frac{1}{2}\xi' Q\xi - \int_{\mathbb{R}^d} \left(e^{i\xi' y} - 1 - i\xi' y \mathbb{1}_{\{\|y\|<1\}}(y) \right) N(dy), \ \xi \in \mathbb{R}^d. \tag{1.2}$$

In Behme and Schnurr (2015) we derived an integral criterion for invariant measures of Itô processes based on the symbol. In that article, we had to use the above formula without the stopping time as obviously global properties might be destroyed by stopping, in contrast to local path properties. This resulted in the fact that for the method proposed in Behme and Schnurr (2015) the symbol had to satisfy a certain growth condition corresponding to bounded semimartingale characteristics of the treated processes and hence to bounded coefficients of the SDEs in the background. Since this is a

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serious restriction, in the present paper we aim to show an alternative to the classical symbol, which then can be used in cases without bounded characteristics/coefficients as long as the corresponding processes have a lower or upper bound (e.g. are restricted to be positive in each component). Hereby, we take up the classic idea of using Laplace transforms instead of Fourier transforms/characteristic functions and define a so called Laplace symbol. In the present paper we emphasize the applicability of the new concept by considering stationary distributions. However, we believe that it can be beneficial in other areas as well.

Thus, after setting the stage in Section 2, we will define the Laplace symbol in Section 3 which also includes several results on its computation as well as on its relation to the generator of the stochastic process. Section 4 is devoted to the derivation of an integral criterion for invariant measures of the underlying processes which relies on the Laplace symbol. Some examples are added to exhibit the usability of the derived criterion.

2. Preliminaries

In this paper we will focus on the class of Itô processes as defined below.

Definition 2.1. An *Itô process* $(X_t)_{t\geq 0}$ is a strong Markov process, which is a semimartingale with respect to every $\mathbb{P}^x$ having semimartingale characteristics of the form

$$\begin{aligned} B_t^{(j)}(\omega) &= \int_0^t \ell^{(j)}(X_s(\omega))\,\mathrm{d}s, \qquad & j=1,\ldots,d, \\ C_t^{jk}(\omega) &= \int_0^t Q^{jk}(X_s(\omega))\,\mathrm{d}s, \qquad & j,k=1,\ldots,d, \\ \nu(\omega;\mathrm{d}s,\mathrm{d}y) &= N(X_s(\omega),\mathrm{d}y)\,\mathrm{d}s, \end{aligned} \tag{2.1}$$

for every $x\in\mathbb{R}^d$ with respect to a fixed cut-off function χ. Here $\ell(x)=(\ell^{(1)}(x),\ldots,\ell^{(d)}(x))'$ is a vector in $\mathbb{R}^d$, $Q(x)$ is a positive semi-definite matrix and N is a Borel transition kernel such that $N(x,\{0\})=0$. We call ℓ, Q and $n:=\int_{y\neq 0}(1\wedge\|y\|^2)\,N(\cdot,\mathrm{d}y)$ the *differential characteristics* of the process.

Itô processes have been characterized in Cinlar and Jacod (1981) as the set of solutions of very general stochastic differential equations (SDEs) of Skorokhod-type. In particular this class includes Lévy processes, solutions of Lévy driven SDEs and Feller processes with sufficiently rich domain. Note that Itô processes in the sense of Definition 2.1 are sometimes in the literature called *Lévy type processes* (cf. Böttcher et al., 2013).

Continuity of the differential characteristics of the treated Itô processes is always sufficient for our purposes. However, we use the concept of fine continuity in order to derive even more general results. Loosely speaking, the advantage of fine continuity is that one has to care about continuity only as far as the process can detect it. We will use fine continuity in a way that is governed by the subsequent result which was established in Blumenthal and Getoor (1968, Thm. II.4.8).

Proposition 2.2. *Let X be a Markov process and $f:\mathbb{R}^d\to\mathbb{R}$ be a Borel-measurable function. Then f is finely continuous if and only if the function $t\mapsto f(X_t)=f\circ X_t$ is right continuous at zero $\mathbb{P}^x$-a.s. for every $x\in\mathbb{R}^d$.*

This proposition yields another advantage of fine continuity: Consider a process $(X_t)_{t\geq 0}$ which is defined on any measurable subset B of $\mathbb{R}^d$ or $\mathbb{R}^d_+$. If a function is finely continuous on the state space B, then we can always extend the process and the function to $\mathbb{R}^d$ or $\mathbb{R}^d_+$ by setting $X_t=x$ for $x\notin B$ and $t\geq 0$. The resulting function is again finely continuous.

3. The Laplace symbol

3.1. Definition of the Laplace symbol

The Laplace symbol can be seen as a state-space dependent right hand side derivative at zero of the Laplace transform of a stochastic process. Since the Laplace transform characterizes the distribution of a random variable, the Laplace symbol in a certain way reflects the infinitesimal changes in the distribution over time.

Definition 3.1. Let $(X_t)_{t\geq 0}$ be a Markov process in $\mathbb{R}^d_+$. Define for every $x,\xi\in\mathbb{R}^d_+$

$$\lambda(x,\xi):=-\lim_{t\downarrow 0}h_\xi(x,t):=-\lim_{t\downarrow 0}\frac{\mathbb{E}^x e^{-(X_t-x)'\xi}-1}{t}. \tag{3.1}$$

Then we call $\lambda:\mathbb{R}^d_+\times\mathbb{R}^d_+\to\mathbb{R}$ the *Laplace symbol* of X with domain $D_\lambda\subseteq\mathbb{R}^d_+\times\mathbb{R}^d_+$ whenever the limit in (3.1) exists for every $x,\xi\in D_\lambda$.

Remark 3.2. Obviously the Laplace symbol as in Definition 3.1 can also be defined for any Markov process which is componentwise bounded from below. Throughout this article we use the bound 0 to ease notation. Similarly, for Markov processes bounded from above, one may define a one-sided symbol $\lambda_-:\mathbb{R}^d_-\times\mathbb{R}^d_-\to\mathbb{R}$.

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