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Gram–Charlier- like expansions of power-raised hyperbolic secant laws

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ABSTRACT

Spherical distributions arise quite naturally as multivariate versions of univariate (even) densities and prove useful in several applications. Likewise their univariate counterparts, they may not always meet the kurtosis requirements of empirical evidence. This paper devises a methodological approach which duly reshapes spherical distributions to match kurtosis requirements to due extent. This approach is tailored to the family of power-raised hyperbolic secant laws and hinges on Gram–Charlier-like expansions via second-degree orthogonal polynomials.

1. Introduction

There is ample evidence that in several fields of economic interest empirical distributions are heavy tailed. Indeed, the occurrence of excess kurtosis is well acknowledged in financial literature (see for example Szego, 2004 and the reference quoted therein). To meet possibly severe kurtosis requirements one strand of research refereed to well-behaved distributions other than Gaussian (Mills and Markellos, 2008; Rachev et al., 2010). A second and more recent stream (Zoia, 2010; Faliva et al., 2016) devised families of distributions in the form of Gram–Charlier-like expansions (GCL). This paper develops further this second approach by moving from the univariate to the multivariate case on a spherical distribution argument (Cambanis et al., 1981). Gram–Charlier-like expansions of spherical hyperbolic secant, logistic and Gaussian laws are provided via properly designed orthogonal polynomials. A novel intriguing relationship between the Mellin transform and the generalized hypergeometric function clears the way to obtain closed-form expressions for the moments and parameters of GCL in terms of classical special functions.

2. Spherical distributions and their moments

The class of spherical distributions corresponds to the class of rotationally symmetric distributions (see e.g., Cambanis et al., 1981; Fang and Zhang, 1990; Gomez et al., 2003). Should an n-dimensional random vector $\boldsymbol{x}$ have a spherical representation, then the following would apply

$$\boldsymbol{x} = RU \tag{2.1}$$

where $R = (\boldsymbol{x}'\boldsymbol{x})^{1/2}$ is a positive random variable, known as generating variate, independent of U, which is uniformly distributed on the unit hypersphere. The density of $\boldsymbol{x}$, $\widetilde{g}_n(\boldsymbol{x})$ hereafter, can be specified as

$$\widetilde{g}_n(\boldsymbol{x}) = k(\boldsymbol{x}'\boldsymbol{x})^{(1-n)/2} f_R((\boldsymbol{x}'\boldsymbol{x})^{1/2}), \quad k = 2^{-1}(\pi)^{-n/2}\Gamma(n/2), \tag{2.2}$$

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where $\Gamma(.)$ is the Euler gamma function and f_R is the density of R, which can be written as

$$f_R(r) = 2M(n)^{-1}r^{n-1}g\left(r^2\right), \quad n > 0. \tag{2.3}$$

Here $g(.)$ is a non-negative Lebesgue measurable function, called density generator, whose Mellin transform

$$M(n) = 2\int_0^\infty r^{n-1}g(r^2)dr \tag{2.4}$$

is finite. Simple computations prove that the jth moment, m_j, of the generating variate R, which according to (2.2) affects the shape of the spherical variable, can be expressed as follows (2.5)

$$m_j = \int_0^\infty r^j f_R(r)\,dr = M(n+j)/M(n) \quad j = 1. \tag{2.5}$$

According to Mardia (1970), the expression $K(n) = [(\boldsymbol{X}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})]^2$, which can be used to measure the kurtosis of a n-variate random vector $\boldsymbol{X}$ with μ as vector mean and $\boldsymbol{\Sigma}$ variance–covariance matrix, in a spherical context ($\boldsymbol{\mu} = \boldsymbol{0}$ and $\boldsymbol{\Sigma} = n^{-1}E(R^2)\boldsymbol{I}_n$), becomes $n^2E(R^4)(E(R^2))^{-2}$ (see e.g. Gomez et al., p. 349, Th 4. (2003) and Zografos, (2008)) and, accordingly, in light of (2.5) and (2.4) can be written as

$$K(n) = n^2 m_4 m_2^{-2} = n^2 M(n+4)\,M(n)\,M(n+2)^{-2}. \tag{2.6}$$

This expression is particularly informative as it directly links this index to the moments of the modular variable, which will play an important role in the following analysis. Given this premise, let us move to specify a spherical distribution starting from a given univariate symmetric density, which in our case will be a member of the Power-raised Hyperbolic Secant (PHS) family. The PHS's are bell-shaped distributions, which originate from the hyperbolic secant law raised to a positive power λ (Faliva and Zoia, 2017). In standard form, a PHS density is specified as follows

$$f_\lambda(x) = b(sech(ax))^\lambda, \lambda > 0, a = 2^{-1/2}\psi^{(1)}(\lambda/2)^{(1/2)}, \qquad b = aB^{-1}\left(\lambda/2, \quad 2^{-1}\right) \tag{2.7}$$

with $\psi^{(1)}(.)$ and $B(.,.)$ denoting the trigamma and beta function, respectively (see, e.g., Davis, 1965). Noteworthy PHS distributions are the hyperbolic secant, logistic and Gaussian laws, corresponding to $\lambda = 1, \lambda = 2$ and $\lambda \to \infty$, respectively. The kurtosis $K(\lambda)$ of a PHS density is given by

$$K(\lambda) = 3 + 2^{-1}\left(\psi^{(3)}(\lambda/2)\right)\left(\psi^{(1)}(\lambda/2)\right)^{-2} \tag{2.8}$$

where $\psi^{(3)}(.)$ is the pentagamma function. $K(\lambda)$ tends to six as $\lambda \to 0$, and to 3 as $\lambda \to \infty$.

Spherical distributions corresponding to PHS densities, SPHS hereafter, originate from density generator of the form

$$g_\lambda(y) = \left(sech\left(ay^{1/2}\right)\right)^\lambda, \quad a > 0, \lambda > 0, y > 0. \tag{2.9}$$

In this connection we have the following

Theorem 2.1. *The density of an n-dimensional SPHS distribution is given by*

$$\widetilde{g}_{n,\lambda}(\boldsymbol{x}) = \frac{1}{2}a^n\frac{\Gamma(n/2)}{(\pi)^{n/2}M_\lambda(n)}\left(sech\left(a(\boldsymbol{x}'\boldsymbol{x})^{1/2}\right)\right)^\lambda, \quad \lambda > 0 \tag{2.10}$$

where $M_\lambda(n)$ is the Mellin transform of $(sech\,\upsilon)^\lambda$ with $\upsilon = ar$ and $r = (\boldsymbol{x}'\boldsymbol{x})^{1/2}$.

The moment $m_{j(n,\lambda)}$ of jth order of its generating variate and the kurtosis are given by

$$m_{j(n,\lambda)} = a^{-j}M_\lambda(n+j)(M_\lambda(n))^{-1} \tag{2.11}$$

$$K_\lambda(n) = n^2M_\lambda(n+4)M_\lambda(n)(M_\lambda(n+2))^{-2}. \tag{2.12}$$

Proof. Expression (2.10) ensues from (2.2) and (2.3) by specifying the density generator as in (2.9) and making use of the Mellin transform and its properties (see, e.g., Zayed, Ch. 10, 1996). Expressions (2.11) and (2.12) follow from (2.6) and (2.7). □

Noteworthy expressions for the moments of the generating variate can be obtained by moving from the Mellin transform to the generalized hypergeometric function. In this connection, let us first establish the following

Lemma 2.2. *The following functional relation holds*

$$(a\lambda)^{-h}2^{\lambda+1}(h-1)!_{h+1}F_h(\lambda, \left[\frac{\lambda}{2}\right]_h; \left[\frac{\lambda}{2}+1\right]_h; -1) = M_\lambda(h) \tag{2.13}$$

where ${}_{h+1}F_h(, ; ;)$ is the generalized hypergeometric function evaluated at $z = -1$ with $h+1$ numerator parameters and h denominator parameters. Here λ is a scalar, whereas $\left[\frac{\lambda}{2}\right]_h$ and $\left[\frac{\lambda}{2}+1\right]_h$ are vectors whose h components are all equal to $\lambda/2$ and $\lambda/2+1$, respectively.