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Intermittency of trawl processes

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ABSTRACT

We study the limiting behavior of continuous time trawl processes which are defined using an infinitely divisible random measure of a time dependent set. In this way one is able to define separately the marginal distribution and the dependence structure. One can have long-range dependence or short-range dependence by choosing the time set accordingly. We introduce the scaling function of the integrated process and show that its behavior displays intermittency, a phenomenon associated with an unusual behavior of moments.

1. Introduction

Trawl processes form a class of stationary infinitely divisible processes that allow the marginal distribution and dependence structure to be modeled independently from each other (see Barndorff-Nielsen (2011) and Barndorff-Nielsen et al. (2014, 2015)). They are defined by

$$X(t) = \Lambda(A_t), \quad t \in \mathbb{R}, \tag{1}$$

where Λ is a homogeneous infinitely divisible independently scattered random measure (*Lévy basis*) and $A_t = A + (0, t)$ for some Borel subset A of $\mathbb{R} \times \mathbb{R}$ of finite Lebesgue measure. The set A is called the *trawl* and is usually specified using the *trawl function* $g : [0, \infty) \to [0, \infty)$ as

$$A = \{(\xi, s) : 0 \leq \xi \leq g(-s),\ s \leq 0\},$$

so that

$$A_t = \{(\xi, s) : 0 \leq \xi \leq g(t - s),\ s \leq t\}.$$

As explained in Barndorff-Nielsen et al. (2015) the trawl A can be regarded as a fishing net dragged along the sea, so that at time t it is in position A_t. A similar structure can be found in Wolpert and Taqqu (2005). To any Lévy basis Λ there corresponds a Lévy process $L = \{L(t),\ t \geq 0\}$ referred to as the *Lévy seed*. The choice of the Lévy seed determines the marginal law of the trawl process, while the shape of the trawl set A controls the dependence structure. In particular, taking the trawl function to be $-(\alpha + 1)$-regularly varying at infinity for some $\alpha \in (0, 1)$, one obtains long-range dependence of the resulting trawl process. See Section 2 for details.

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A discrete time analog of the trawl process (1) has been defined in Doukhan et al. (2016) as a process

$$Y(k)=\sum_{j=0}^{\infty} Z^{(k-j)}(a_j), \quad k \in \mathbb{Z}, \tag{2}$$

where $Z^{(k)}=\{Z^{(k)}(u),\ u \in \mathbb{R}\}$, $k \in \mathbb{Z}$ are i.i.d. copies of some process $Z=\{Z(u),\ u \in \mathbb{R}\}$ stochastically continuous at zero and $(a_j)_{j\in\mathbb{N}}$ is a sequence of constants such that $a_j \to 0$ as $j \to \infty$. The long-range dependent case in the discrete time setting corresponds to choosing a sequence $a_j = L(j) j^{-\alpha-1}$ where L is some slowly varying function.

The correspondence of $Y(k)$ in (2) with the continuous time trawl process (1) is the following. Suppose on one hand that $\{Y_k,\ k \in \mathbb{Z}\}$ is a discrete time trawl process with trawl sequence $(a_j)_{j\in\mathbb{N}_0}$ and such that Z is some two-sided Lévy process $L=\{L(t),\ t\in\mathbb{R}\}$. On the other hand, let $\{X(t),\ t \in \mathbb{R}\}$ be a trawl process with Lévy seed process L and trawl specified by the function

$$g(x)=\sum_{j=0}^{\infty} a_j \mathbf{1}_{(-j-1,-j]}(x).$$

Then $\{Y(k),\ k \in \mathbb{Z}\}$ is equal in law to a discretized process $\{X(k),\ k \in \mathbb{Z}\}$ (Doukhan et al., 2016). While the marginal distribution of the trawl process $X(t)$ in (1) is necessarily infinitely divisible, the discrete time setting allows for rather general seed processes.

An important and interesting question regarding trawl processes are limit theorems for cumulative processes arising from them. Assuming the trawl process has zero mean, in the discrete time setup, the cumulative process would be a partial sum process

$$S_n(t)=\sum_{k=1}^{[nt]} Y(k),$$

while in the continuous time it is natural to consider the integrated process

$$X^*(t)=\int_0^t X(u)du.$$

However, as we show in this paper, the corresponding limiting behavior of moments seems to be unexpected.

Doukhan et al. (2016) have interesting results. In their paper, a limit theorem is proved with convergence to fractional Brownian motion for the partial sum process formed from a zero mean long-range dependent discrete time trawl process (Doukhan et al., 2016, Theorem 1(i)). The crucial condition for this result is the following small time moment asymptotics of the seed process: for some $\delta > 0$, one has

$$\mathbb{E}|Z(t)|^{2+\delta} = O(|t|^{\frac{2+\delta}{2}}), \quad \text{as } t \to 0. \tag{3}$$

One may wonder whether the proof of Doukhan et al. (2016, Theorem 1(i)) could be extended to the continuous time trawl processes. The following argument shows that the condition (3) excludes the possibility that the seed process is any Lévy process except Brownian motion. Indeed, suppose Z is a Lévy process with Lévy measure ν such that $\mathbb{E}Z(1)=0$. By Asmussen and Rosiński (2001, Lemma 3.1) for any $\delta \geq 0$ such that $\mathbb{E}|Z(1)|^{2+\delta} < \infty$, one has

$$\lim_{n\to\infty} n\mathbb{E}|Z(1/n)|^{2+\delta} = \int_{\mathbb{R}} |x|^{2+\delta}\nu(dx).$$

Hence, $\mathbb{E}|Z(t)|^{2+\delta} \sim C_\delta t$ as $t \to 0$ for any $\delta > 0$ and (3) cannot hold unless $\nu = 0$ and Z is a Brownian motion. Since Brownian motion is self-similar with self-similarity parameter 1/2, condition (3) holds for Brownian motion but not for any other Lévy process. Hence, the conditions of Doukhan et al. (2016, Theorem 1(i)) cannot be adapted to obtain a limit theorem for a continuous time trawl process (1) when generated by a non-Gaussian seed process.

Our focus in this paper is on the convergence of moments. We prove that the integrated long-range dependent non-Gaussian trawl processes satisfying certain regularity assumptions on the trawl, have a specific limiting behavior called *intermittency*. A precise definition is given in Section 3. Such a property has so far been established for a partial sum and integrated process of superpositions of Ornstein–Uhlenbeck type processes (see Grahovac et al. (2016, 2017)). This result sheds a new light on the limiting behavior related to trawl processes.

2. Trawl processes

In this section we define trawl processes following Barndorff-Nielsen (2011) and Barndorff-Nielsen et al. (2014, 2015).