STAPRO: 8130 Model 3G pp. 1–7 (col. fig: NIL)

ARTICLE IN PRESS

Statistics and Probability Letters xx (xxxx) xxx–xxx

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Stein's lemma for truncated elliptical random vectors

Tomer Shushi

Actuarial Research Center, Department of Statistics, University of Haifa, Israel
Department of Economics and Business Administration, Ariel University, Israel

ARTICLE INFO

Article history:
Received 13 September 2017
Received in revised form 29 January 2018
Accepted 9 February 2018
Available online xxxx

Keywords:
Capital asset pricing models
Density generator
Elliptical distributions
Siegel's formula
Stein's lemma
Truncated random vectors

ABSTRACT

In this letter we derive the multivariate Stein's lemma for truncated elliptical random vectors. The results in this letter generalize Stein's lemma for elliptical random vectors given in Landsman and Nešlehová (2008), and the tail Stein's lemma given in Landsman and Valdez (2016). We give a conditional Stein's-type inequalities and a conditional version of Siegel's formula for the elliptical distributions, and by that we generalize results obtained in Landsman et al. (2013) and in Landsman et al. (2015). Furthermore, we show applications of the main results in the letter for risk theory.

1. Introduction

Stein's lemma (Stein, 1981) gives an important and elegant formula for the multivariate normal distributions, and has many applications in quantitative finance and statistics (Froot, 2007; Landsman and Nešlehová, 2008; Adcock, 2014; Gron et al., 2012; Vanduffel and Yao, 2017). In quantitative finance, this lemma is used to calculate capital asset pricing models for returns of arbitrary number of dependent assets (Fama and French, 2004; Levy, 2012; Barberis et al., 2015). Furthermore, a vast number of models deal with asset returns as truncated random variables. For example, value at risk measure, tail value at risk measure, truncated regression models, and censored quantile regressions, are models that are based on truncated random variables (Liu, 1994; Cousin and Di Bernardino, 2014; Landsman et al., 2016; Kong and Xia, 2017). Therefore, it seems natural to generalize Stein's lemma for truncated random vectors.

Let $(X_1, X_2)^T$ be a bivariate normal random vector and consider a differentiable function $h : \mathbb{R}^2 \to \mathbb{R}$ such that $E\left(\left|h'(X_1)\right|\right) < \infty$. Then, the bivariate Stein's lemma states that (see, for instance, Landsman and Nešlehová, 2008)

$$Cov\,(h(X_1), X_2) = Cov\,(X_1, X_2) \cdot E\left(h'(X_1)\right).$$

Suppose we have an n-variate normal random vector $\mathbf{X} \backsim N_n(\mu, \Sigma)$, and a differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ such that the expectation of the norm of $\nabla h(\mathbf{X}) = \left(\partial h(\mathbf{x})/\partial x_1, \partial h(\mathbf{x})/\partial x_2, \ldots, \partial h(\mathbf{x})/\partial x_n\right)^T$ exists. Then, Stein's lemma is given by Stein (1981)

$$Cov\,(h(\mathbf{X}), \mathbf{X}) = \Sigma E\,(\nabla h(\mathbf{X})).$$

In the following section we give a definition for the family of elliptical distributions and describe several of its important properties. In Section 3 we derive Stein's lemma for truncated elliptical random vectors, we then generalize several results about Stein's identity and Siegel's formula, and show important inequalities of the proposed truncated Stein's lemma. Section 4 presents applications of the main results in the letter for risk theory.

E-mail address: tomershushi@gmail.com.

https://doi.org/10.1016/j.spl.2018.02.008

2. The elliptical distributions

The family of elliptical distributions is an extension of the normal distribution (Cambanis et al., 1981) into a broader family. Let $\mathbf{X}$ be $n \times 1$ random vector following elliptical distribution, $\mathbf{X} \backsim E_n(\mu, \Sigma, g_n)$. Then, the probability density function (pdf) of $\mathbf{X}$ is

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{|\Sigma|}} g_n\left(\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\right), \mathbf{x} \in \mathbb{R}, \tag{1}$$

where $g_n(u)$, $u \geq 0$, is called the density generator of $\mathbf{X}$, μ is an $n \times 1$ vector of means, and Σ is an $n \times n$ scale matrix. The characteristic function of $\mathbf{X}$ takes the form $\varphi_{\mathbf{X}}(\mathbf{t}) = \exp(i\mathbf{t}^T\mu)\psi(\frac{1}{2}\mathbf{t}^T\Sigma\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^n$, with some function $\psi(u) : [0, \infty) \to \mathbb{R}$, called the characteristic generator. The covariance matrix of $\mathbf{X}$ is then given by $Cov(\mathbf{X}) = \frac{\sigma_Z^2}{n}\Sigma$, where $\sigma_Z^2 = -\psi'(0)$. For the sequel, we define a cumulative generator function $\overline{G}_n(u)$ (see, for instance, Landsman et al. (2016)), such that

$$\overline{G}_n(u) = \int_u^{\infty} g_n(x)\,dx, \tag{2}$$

and an associated elliptical random vector $\mathbf{X}^* \backsim E_n(\mu, \Sigma, n\overline{G}_n/\sigma_Z^2)$ whose pdf takes the form

$$f_{\mathbf{X}^*}(\mathbf{t}) = \frac{n}{\sigma_Z^2\sqrt{|\Sigma|}}\overline{G}_n\left(\frac{1}{2}(\mathbf{t}-\mu)^T\Sigma^{-1}(\mathbf{t}-\mu)\right), \mathbf{t} \in \mathbb{R}^n. \tag{3}$$

3. Stein's lemma for truncated elliptical random vectors

We define an almost differentiable function $h : \mathbb{R}^m \to \mathbb{R}$, $1 \leq m < n$, under the following condition

$$E\left(\left\|\nabla h\left(\mathbf{X}^*\right)\right\|\right) < \infty, \tag{4}$$

where $\nabla = d/d\mathbf{x}$ is the n-multivariate operator of first derivatives, and $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^m$.

To present Stein's lemma for truncated elliptical random vectors, we define a subset of $\mathbb{R}^n$, $R \subseteq \mathbb{R}^n$ which is a subset of all possible outcomes of $\mathbf{X} \in \mathbb{R}^n$, and a conditional expected value $E^{\mathcal{R}}(h(\mathbf{X})(\mathbf{X}-\mu)) := E(h(\mathbf{X})(\mathbf{X}-\mu)|\mathbf{X} \in \mathcal{R})$ with the conditional covariance of $\left(h(\mathbf{X}), \mathbf{X}^T\right)^T$, $Cov^{\mathcal{R}}(h(\mathbf{X}), \mathbf{X}) := E^{\mathcal{R}}(h(\mathbf{X})(\mathbf{X}-\mu)) - E(h(\mathbf{X})|\mathbf{X} \in \mathcal{R}) \cdot E(\mathbf{X}-\mu|\mathbf{X} \in \mathcal{R})$.

Theorem 1. *Let $\mathbf{X}$ be an elliptical random vector, $\mathbf{X} \backsim E_n(\mu, \Sigma, g_n)$. Then, Stein's lemma for the truncated random vector $\mathbf{X}|(\mathbf{X} \in \mathcal{R})$ takes the form*

$$E^{\mathcal{R}}(h(\mathbf{X})(\mathbf{X}-\mu)) = Cov(\mathbf{X})E\left(\nabla h\left(\mathbf{X}^*\right)|\mathbf{X}^* \in \mathcal{R}\right)\frac{F^*(\mathcal{R})}{F(\mathcal{R})} - E^{\delta}\left(h\left(\mathbf{X}^*\right)\right). \tag{5}$$

Here

$$E^{\delta}\left(h\left(\mathbf{X}^*\right)\right) = E\left(h\left(\mathbf{X}^*\right)\delta\left(\mathbf{X}^* \in \mathcal{R}\right)\right)\frac{\sigma_Z}{\sqrt{n}F(\mathcal{R})},$$

where

$$\delta\left(\mathbf{X}^* \in \mathcal{R}\right) = \begin{pmatrix}\delta_{Cov(\mathbf{X})_1^{1/2}} & \delta_{Cov(\mathbf{X})_2^{1/2}} & \cdots & \delta_{Cov(\mathbf{X})_n^{1/2}}\end{pmatrix} \tag{6}$$

is an $n \times 1$ vector of surface delta functions $\delta_{\mathbf{a}} = -\mathbf{a} \cdot \nabla 1_{\mathbf{X}^* \in \mathcal{R}}$ (see, Lange (2012)), $Cov(\mathbf{X})_i^{1/2}$, $i = 1, 2, \ldots, n$ is the ith row of $Cov(\mathbf{X})^{1/2}$, $F(\mathcal{R}) = \Pr(\mathbf{X} \in \mathcal{R})$, and $F^*(\mathcal{R}) = \Pr(\mathbf{X}^* \in \mathcal{R})$.

Proof. Using the indicator function $1_{\mathbf{x} \in \mathcal{R}}$ and under the linear transformation $\Sigma^{-1/2}(\mathbf{X}-\mu) = \mathbf{z}$, we have

$$\begin{aligned}&E(h(\mathbf{X})(\mathbf{X}-\mu)|\mathbf{X} \in \mathcal{R})\\ &= \frac{1}{|\Sigma|^{1/2}F(\mathcal{R})}\int_{\mathcal{R}} h(\mathbf{x})(\mathbf{X}-\mu)g_n\left(\frac{1}{2}(\mathbf{X}-\mu)^T\Sigma^{-1}(\mathbf{X}-\mu)\right)d\mathbf{x}\\ &= F(\mathcal{R})^{-1}\Sigma^{1/2}\int_{\mathbb{R}^n} h\left(\mu + \Sigma^{1/2}\mathbf{z}\right)1_{\mathbf{z} \in \mathcal{R}_Z} \cdot \mathbf{z}g_n\left(\frac{1}{2}\mathbf{z}^T\mathbf{z}\right)d\mathbf{z},\end{aligned}$$

where the set $\mathcal{R}_Z$ is such that $\{\mathbf{X} \in \mathcal{R}\} = \{\mathbf{Z} \in \mathcal{R}_Z\}$. Taking into account the density generator $\overline{G}(u)$ (2) and the associated pdf (3), similar to Proposition 2 in Landsman et al. (2015) after partial integration and algebraic calculations, we observe that

$$\begin{aligned}E(h(\mathbf{X})(\mathbf{X}-\mu)|\mathbf{X} \in \mathcal{R}) = Cov(\mathbf{X})E\left(\nabla h\left(\mathbf{X}^*\right)|\mathbf{X}^* \in \mathcal{R}\right)\frac{F^*(\mathcal{R})}{F(\mathcal{R})}\\ + \frac{\sigma_Z}{\sqrt{n}F(\mathcal{R})}Cov(\mathbf{X})^{1/2}E\left(h\left(\mathbf{X}^*\right)\nabla 1_{\mathbf{X}^* \in \mathcal{R}}\right).\end{aligned}$$