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Truncated fractional moments of stable laws

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ABSTRACT

Expressions are given for the truncated fractional moments $E(X-a)_+^p$ of a general stable law. These involve families of special functions that arose out of the study of multivariate stable densities and probabilities. This main result yields expressions for $E(X-a)_+$ when the index of stability $\alpha > 1$, and expressions for fractional absolute and signed moments for any stable law.

1. Introduction

A univariate stable r.v. Z with index α, skewness β, scale γ, and location δ has characteristic function

$$\phi(u) = \phi(u|\alpha,\beta,\gamma,\delta) = E\exp(iuZ) = \exp(-\gamma^\alpha[|u|^\alpha + i\beta\eta(u,\alpha)] + iu\delta), \tag{1}$$

where $0 < \alpha \le 2$, $-1 \le \beta \le 1$, $\gamma > 0$, $\delta \in \mathbb{R}$ and

$$\eta(u,\alpha) = \begin{cases} -(\operatorname{sign} u)\tan(\pi\alpha/2)|u|^\alpha & \alpha \ne 1 \\ (2/\pi)\, u \log|u| & \alpha = 1. \end{cases}$$

In the notation of Samorodnitsky and Taqqu (1994), this is a $S_\alpha(\gamma,\beta,\delta)$ distribution. We will use the notation $X \sim \mathbf{S}(\alpha,\beta,\gamma,\delta;1)$ (the ";1" is used to distinguish between this parameterization and a continuous one used below). A stable r.v. is strictly stable when ($\alpha \ne 1$ and $\delta = 0$) or when ($\alpha = 1$ and $\beta = 0$).

The purpose of this paper is to derive expressions for truncated fractional moments $E(X-a)_+^p = E((X-a)\mathbb{1}_{\{X>a\}})^p$ for general $X \sim \mathbf{S}(\alpha,\beta,\gamma,\delta;1)$. We first note the simple result that $E(X-a)_+^p = EY_+^p$ where $Y = X-a \sim \mathbf{S}(\alpha,\beta,\gamma,\delta-a;1)$, so an expression for fractional moments truncated at zero for general stable Y gives an expression for an arbitrary truncation point. Hence, in what follows we will focus on truncation at zero, but since we allow an arbitrary shift, the formulas derived give $E(X-a)_+^p$ for any a.

Most of the known results are for X strictly stable and the truncation is at zero. Hardin (1984) gave an expression for $E|X|^p$ in the strictly stable case; see Corollary 3 for the statement. Zolotarev (1986) gives an expression for EX_+^p when X is strictly stable, and section 8.3 of Paolella (2007) gives two derivations in the strictly stable case. When $1 < p < \alpha$, Matsui and Pawlas (2016) give expressions for $E|X-a|^p$ when $\alpha > 1$, $\beta = 0$ and $1 < p < \alpha$. In Theorem 1 expressions for truncated moments of stable laws are given for any $\alpha \in (0,2)$, any $\beta \in [-1,1]$, any scale $\gamma > 0$, and any shift δ. That easily leads to expressions for $E|X|^p$ and related signed moments.

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Truncated moments are of interest in finance and insurance: when X models the loss of an asset, $E(X-a)_+$ is the expected loss given $X > a$. They also arise in the stable GARCH model of Mittnik et al. (2002). Fractional moments can be used to estimate parameters, see Section 5.7 of Nikias and Shao (1995). And Section 2.6 of Zolotarev (1986) uses $E|X|^p$ to characterize the distribution of $|X|$ via the Mellin transform.

To give expressions for the truncated moment, define the functions for real x and d

$$g_d(x|\alpha,\beta) = \begin{cases} \int_0^\infty \cos(xr+\beta\eta(r,\alpha))\, r^{d-1} e^{-r^\alpha}\, dr & 0<d<\infty \\ \int_0^\infty [\cos(xr+\beta\eta(r,\alpha))-1]\, r^{d-1}\, e^{-r^\alpha}\, dr & -2\min(1,\alpha)<d\le 0, \end{cases}$$

$$\widetilde{g}_d(x|\alpha,\beta) = \begin{cases} \int_0^\infty \sin(xr+\beta\eta(r,\alpha))\, r^{d-1} e^{-r^\alpha}\, dr & -\min(1,\alpha)<d<\infty \\ \int_0^\infty [\sin(xr+\beta\eta(r,\alpha))-xr]\, r^{d-1} e^{-r^\alpha}\, dr & \alpha>1, -\alpha<d\le -1. \end{cases}$$

The functions $g_d(\cdot|\alpha,\beta)$ and $\widetilde{g}_d(\cdot|\alpha,\beta)$, for integer subscripts $d = 1, 2, 3, \ldots$ were introduced in Abdul-Hamid and Nolan (1998). (The notation was slightly different there: a factor of $(2\pi)^{-d}$ was included in the definition and $g_{\alpha,d}(x,\beta)$ was used instead of $g_d(x|\alpha,\beta)$, while $q_{\alpha,d}(x,\beta)$ was used instead of $\widetilde{g}_d(x|\alpha,\beta)$.)

The expressions for EX_+^p will involve the functions $g_{-p}(\cdot|\alpha,\beta)$ and $\widetilde{g}_{-p}(\cdot|\alpha,\beta)$, i.e. negative fractional values of the subscript d. Before proving that result, we show that the functions $g_d(\cdot|\alpha,\beta)$ and $\widetilde{g}_d(\cdot|\alpha,\beta)$ have multiple uses. For a standardized, i.e. scale $\gamma = 1$ and location $\delta = 0$, univariate stable law, Fourier inversion of the characteristic function shows that the d.f. and density are given by

$$F(x|\alpha,\beta) - F(0|\alpha,\beta) = \frac{1}{\pi}\left(\widetilde{g}_0(x|\alpha,\beta) - \widetilde{g}_0(0|\alpha,\beta)\right) \tag{2}$$
$$f(x|\alpha,\beta) = \frac{1}{\pi} g_1(x|\alpha,\beta).$$

We note that there are explicit formulas for $F(0|\alpha,\beta)$ when $\alpha \ne 1$.

The $g_d(\cdot|\alpha,\beta)$ functions are used in a similar way to give d-dimensional stable densities, see Theorem 1 of Abdul-Hamid and Nolan (1998) (note that there is a sign mistake in that formula when $\alpha = 1$), and Nolan (2018) uses both $g_d(\cdot|\alpha,\beta)$ and $\widetilde{g}_d(\cdot|\alpha,\beta)$ to give an expression for multivariate stable probabilities. That paper also shows that the conditional expectation $E(X_2|X_1 = x)$ for bivariate stable (X_1, X_2) can be expressed in terms of these functions.

2. Truncated moments EX_+^p

The main result of this paper is the following expression for the fractional truncated moment of a stable r.v. When $p = 0$, EX_+^0 is interpreted as $\int_0^\infty f(x)dx$.

Theorem 1. *Let $X \sim \mathbf{S}(\alpha,\beta,\gamma,\delta;1)$ with any $0<\alpha<2$ and any $-1\le\beta\le 1$ and set*

$$\delta^* = \begin{cases} \delta/\gamma & \alpha \ne 1 \\ \delta/\gamma + \frac{2}{\pi}\beta\log\gamma & \alpha = 1. \end{cases}$$

For $-1<p<\alpha$, define $m^p(\alpha,\beta,\gamma,\delta) = EX_+^p$.
(a) When $-1<p<0$,

$$m^p(\alpha,\beta,\gamma,\delta) = -\gamma^p \frac{\Gamma(p+1)}{\pi}\left[\sin\left(\tfrac{\pi p}{2}\right) g_{-p}(-\delta^*|\alpha,\beta) + \cos\left(\tfrac{\pi p}{2}\right)\widetilde{g}_{-p}(-\delta^*|\alpha,\beta)\right].$$

When $p = 0$,

$$m^0(\alpha,\beta,\gamma,\delta) = P(X>0) = \frac{1}{2} - \frac{1}{\pi}\widetilde{g}_0(-\delta^*|\alpha,\beta).$$

When $0<p<\min(1,\alpha)$,

$$m^p(\alpha,\beta,\gamma,\delta) = \gamma^p \frac{\Gamma(p+1)}{\pi}\left[\sin\left(\tfrac{\pi p}{2}\right)\left(\Gamma(1-p/\alpha)/p - g_{-p}(-\delta^*|\alpha,\beta)\right) - \cos\left(\tfrac{\pi p}{2}\right)\widetilde{g}_{-p}(-\delta^*|\alpha,\beta)\right].$$

When $p = 1 < \alpha < 2$,

$$m^p(\alpha,\beta,\gamma,\delta) = \gamma\left[\frac{\delta^*}{2} + \frac{1}{\pi}\left(\Gamma(1-1/\alpha) - g_{-1}(-\delta^*|\alpha,\beta)\right)\right].$$