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Reduction functions for the variance function of one-parameter natural exponential family

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ABSTRACT

We show that, when a random variable has a parametric distribution as a member of an infinitely divisible natural exponential family whose induced measure is absolutely continuous with respect to its basis measure, there exists a deterministic function, referred to as "reduction function", such that the random variable transformed by this function is an unbiased estimator of the variance of the random variable. Our result can be used in estimating latent structure in high-dimensional data and in implementing iterative reweighted least squares for generalized linear models.

1. Introduction

Let $\mathbb{E}$ and $\mathbb{V}$ be the mean and variance operators. For a real-valued random variable ξ, it is often needed to find a function φ, which we call a "reduction function" for ξ, such that $\mathbb{E}[\varphi(\xi)] = \mathbb{V}[\xi]$, i.e., the transformed random variable $\varphi(\xi)$ is an unbiased estimate of the variance of ξ. For example, Theorem 2, Theorem 5 and Lemma 8 of Chen and Storey (2015) reveal that a reduction function induces a consistent estimator of a linear latent space in high-dimensional data. A second example is the implementation of iterative reweighted least squares (IWLS, Jorgensen, 2006) for generalized linear models (GLMs), where an unbiased estimate of the variance of a response variable ζ is needed and can be set as $\varphi(\zeta)$ if φ is a reduction function for ζ.

It can be perceived that the existence of φ depends crucially on the variance–mean relationship of ξ. For example, when the distribution function of ξ is a member of a natural exponential family with quadratic variance function (NEF-QVF, Morris, 1982), φ exists and is given by Lemma 7 and Table 1 of Chen and Storey (2015). However, φ may not exist at all if ξ has other distribution functions. Unfortunately, it seems to be very hard to determine when a reduction function exists in general. So, we focus on the case where ξ has a parametric distribution in the form of an NEF $\mathcal{F}$. This allows partial but useful results on the existence of φ due to the richness and wide usage of NEFs in probability and statistics (McCullagh and Nelder, 1989; Letac, 1992). We show that φ exists if the NEF $\mathcal{F}$ is infinitely divisible and its induced measure is absolutely continuous with respect to the basis measure of $\mathcal{F}$; see Proposition 1. In view of this, our result also contributes to the classification of NEFs via the existence of reduction functions rather than traditionally from the perspective of if a positive, analytic function can generate an NEF.

The rest of this article is organized as follows. In Section 2 we provide some preliminaries on NEFs, and in Section 3 we present our main results with illustrating examples.

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2. Preliminaries on NEFs

We review the definition of an NEF and some properties of its mean and variance functions, which can be found in Letac (1992). Let β be a positive Radon measure on $\mathbb{R}$ that is not concentrated on one point. Let $L(\theta) = \int e^{x\theta}\beta(dx)$ for $\theta \in \mathbb{R}$ be its Laplace transform and Θ be the maximal open set containing θ such that $L(\theta) < \infty$. Suppose Θ is not empty and let $\kappa(\theta) = \log L(\theta)$ be the cumulant function of β. Then

$$\mathcal{F} = \{F_\theta : F_\theta(dx) = \exp(\theta x - \kappa(\theta))\beta(dx), \theta \in \Theta\}$$

forms an NEF with respect to the basis measure β. For the NEF $\mathcal{F}$, the mapping

$$\mu(\theta) = \int xF_\theta(dx) = \kappa'(\theta) \tag{1}$$

defines the mean function $\mu : \Theta \to U$ with $U = \mu(\Theta)$, and U is called the "mean domain". Further, the mapping

$$V(\theta) = \int (x - \mu(\theta))^2 F_\theta(dx) = \kappa''(\theta) \tag{2}$$

defines the variance function (VF). Let $\theta = \theta(\mu)$ be the inverse function of μ. Then V can be parametrized by μ as

$$V(\mu) = \int (x-\mu)^2 F_{\theta(\mu)}(dx) \text{ for } \mu \in U,$$

and the pair (V, U) is called the VF of $\mathcal{F}$.

Let $\xi_\theta, \theta \in \Theta$ have distribution $F_\theta \in \mathcal{F}$, then its mean is $\mu(\theta)$ and variance $V(\theta)$ as defined by (1) and (2). In this setting, a reduction function φ is a measurable function independent of θ such that

$$\mathbb{E}[\varphi(\xi_\theta)] = \mathbb{V}[\xi_\theta], \quad \forall\theta \in \Theta, \tag{3}$$

i.e.,

$$\int \varphi(x)e^{x\theta}\beta(dx) = L(\theta)\kappa''(\theta), \forall\theta \in \Theta. \tag{4}$$

3. Reduction functions for some infinitely divisible NEFs

Let δ_y be the Dirac mass at $y \in \mathbb{R}$. By Theorem 6.2 on page 12 of Letac (1992) (see also Theorem 3.2 of Kokonendji and Seshadri, 1994), β is infinitely divisible if and only if $\kappa''(\theta) = \int e^{\theta x}\rho(dx)$, where

$$\rho(dx) = \sigma^2\delta_0(dx) + x^2\nu(dx) \tag{5}$$

for some constant $\sigma \geq 0$ and ν is the Lévy measure of β on $\mathbb{R} \setminus \{0\}$ such that

$$\int_{\mathbb{R}\setminus\{0\}} \min\left(1, x^2\right)\nu(dx) < \infty. \tag{6}$$

For two measures ν_1 and ν_2, let $\nu_1 \ll \nu_2$ denote that ν_1 is absolutely continuous with respect to ν_2 and $\nu_1 * \nu_2$ denote their convolution. We provide a sufficient condition on the existence of φ.

Proposition 1. *Assume $\beta \in \mathcal{F}$. Then the following three assertions are equivalent: (i) $\mathcal{F}$ is infinitely divisible; (ii) $\kappa''(\theta) = \int e^{\theta x}\rho(dx)$ for all $\theta \in \Theta$, where ρ is given by (5); (iii) $L(\theta)\kappa''(\theta) = \int e^{\theta x}\alpha(dx)$ for all $\theta \in \Theta$, where $\alpha = \beta * \rho$. Therefore, if $\mathcal{F}$ is infinitely divisible and $\alpha \ll \beta$, then φ exists and is the Radon–Nikodym derivative $\varphi = \frac{d\alpha}{d\beta}$.*

Proof. By Proposition 4.1 on page 9 of Letac (1992), either both β and $\mathcal{F}$ are infinitely divisible or neither are. Obviously, $\int_{\{x\in\mathbb{R}:|x|\geq 1\}} x^{-2}\rho(dx) < \infty$ by (5) and (6). Since $L(\theta) = \int e^{x\theta}\beta(dx)$, setting $\alpha = \beta * \rho$ yields $L(\theta)\kappa''(\theta) = \int e^{\theta x}\alpha(dx)$. If further $\alpha \ll \beta$, then the Radon–Nikodym derivative $\frac{d\alpha}{d\beta}$ is well defined. Let $\varphi = \frac{d\alpha}{d\beta}$. Then $\int \varphi(x)e^{\theta x}\beta(dx) = \int e^{\theta x}\alpha(dx)$ and (3) holds for all $\theta \in \Theta$. Namely, $\varphi = \frac{d\alpha}{d\beta}$ is a reduction function. This completes the proof. ■

We call α in Proposition 1 the "measure induced by the infinitely divisible NEF" $\mathcal{F}$. By Proposition 1, to find φ we first need to check if an NEF $\mathcal{F}$ is infinitely divisible and $\beta \in \mathcal{F}$, and if so, we then check if $\alpha \ll \beta$. For the rest of the article, "reduction function" will be abbreviated as "RF". Let $1_Q(\cdot)$ be the indicator function of a set Q.

Now we investigate the RF for an infinitely divisible NEF whose support is a subset of the set $\mathbb{N}$ of nonnegative integers. Write $\beta = \sum_{n=0}^{\infty}\beta_n\delta_n$ with $\beta_n = \beta(\{n\})$. By the Criterion on page 290 of Feller (1968), β is an infinitely divisible probability measure and concentrated on $\mathbb{N}$ if and only if there exists a nonnegative sequence $\{c_n\}_{n\geq 1}$ such that $\sum_{n=1}^{\infty}c_n z^n$ has radius of convergence $R \geq 1$ and that $\kappa(\theta) = \sum_{n=0}^{\infty}c_n e^{n\theta}$ with $c_0 = 0$ for $\theta \in \Theta = (-\infty, \log R)$. So, the sequence $\{c_n\}_{n\geq 1}$ is given by

$$c_n = n! \lim_{z\to 0}\frac{d^n}{dz^n}\kappa(\log z) \quad \text{for } n \geq 1 \tag{7}$$