Statistics and Probability Letters 135 (2018) 83–91

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Mutual intersection for rough differential systems driven by fractional Brownian motions

Cheng Ouyang [a], Yinghui Shi [b,*], Dongsheng Wu [c]

[a] Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, Chicago, IL 60607, United States
[b] School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, 221116, China
[c] Department of Mathematical Sciences, University of Alabama in Huntsville, Huntsville, AL 35899, United States

ARTICLE INFO

Article history:
Received 28 June 2017
Received in revised form 17 November 2017
Accepted 24 November 2017
Available online 7 December 2017

MSC:
60G15
60H07
60H10

Keywords:
Fractional Brownian motion
Rough differential systems
Mutual intersection

ABSTRACT

Let X^H and X^K be solutions to two stochastic differential equations driven by independent fractional Brownian motions with Hurst parameters H and K, respectively. We study when X^H and X^K intersect with each other over a finite time interval. We also derive Hausdorff and packing dimension results for the set of intersection times of X^H and X^K.

1. Introduction

Random dynamical systems are well established modeling tools for a variety of natural phenomena ranging from physics (fundamental and phenomenological) to chemistry and more recently to biology, economics, engineering sciences and mathematical finance. In many interesting models the lack of any regularity of the external inputs of the differential equation as functions of time is a technical difficulty that hampers their mathematical analysis. The theory of rough paths has been initially developed by T. Lyons (Lyons, 1998) in the 1990's to provide a framework to analyze a large class of driven differential equations and the precise relations between the driving signal and the output (that is the state, as function of time, of the controlled system).

Rough paths theory provides a nice framework to study differential equations driven by Gaussian processes (see Friz and Victoir, 2010a). In particular, using rough paths theory, we may define solutions of stochastic differential equations driven by a fractional Brownian motion. Consider

$$X_t = x + \int_0^t V_0(X_s)ds + \sum_{i=1}^{d} \int_0^t V_i(X_s)dB_s^i, \tag{1}$$

where $x \in \mathbb{R}^n$, $V_0, V_1, \ldots, V_d$ are bounded smooth vector fields on $\mathbb{R}^n$ and $\{B_t, t \geq 0\}$ is a d-dimensional fractional Brownian motion with Hurst parameter $H \in (1/4, 1)$. Existence and uniqueness of solutions to the above equation can be found, for

* Corresponding author.
E-mail addresses: couyang@math.uic.edu (C. Ouyang), shiyinghui@jsnu.edu.cn (Y. Shi), dongsheng.wu@uah.edu (D. Wu).

https://doi.org/10.1016/j.spl.2017.11.012

example, in Lyons and Qian (2002). In particular, when $H = 1/2$, this notion of solution coincides with the solution of the corresponding Stratonovitch stochastic differential equation. It is also clear now (cf. Baudoin and Hairer, 2007; Cass and Friz, 2010; Cass et al., 2013, 2015; Hairer and Pillai, 2013) that under Hörmander's condition the law of the solution X_t has a smooth density $p_t(x, y)$ with respect to the Lebesgue measure on $\mathbb{R}^n$.

In the present work, we are interested in the mutual intersection of two independent solutions to equations of type (1). More precisely, suppose we have two mutually independent fractional Brownian motions $B = (B^1, \ldots, B^d)$ and $\tilde{B} = (\tilde{B}^1, \ldots, \tilde{B}^d)$ from the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with Hurst parameters H and K, respectively. We assume that both H and K are greater than $1/4$, and without loss of generality, assume that $H \leq K$. Note here that we give ourselves the flexibility that the two fractional Brownian motions have different Hurst parameters. Let X^H be the solution to Eq. (1) driven by B. Assume that X^K is the solution to an equation of the same type as (1), but with a different starting point $\tilde{x}$ and possibly another set of vector fields $\tilde{V}_i : i = 0, \ldots, d$, driven by the second fractional Brownian motion $\tilde{B}$. Clearly X^H and X^K are independent. We are interested in when these two solutions intersect with each other over the time interval $[0, 1]$.

The question of mutual intersection as above are usually discussed in the setting of random fields. Standard strategy in solving this problem is to consider the random field of two parameters $Y(s, t) = (X^H_s, X^K_t)$ and translate the question of mutual intersection to the question of hitting probability

$$\text{When do we have } \mathbb{P}\{\, Y \text{ hits } D \text{ on } [0, 1]^2 \,\} > 0\ ?$$

Here $D = \{(x, x) : x \in \mathbb{R}^n\}$ is the diagonal.

The problem about hitting probabilities is important in potential theory of stochastic processes and random fields. Usually, to solve a hitting probability problem, sophisticated computations are expected. We refer to Chen and Zhou (2015), Xiao (2009) and references therein for details.

In this work, we propose a simple approach to the problem, employing current development in the study of Eq. (1). The main idea is described as follows. Since X^H and X^K are independent, we can freeze X^H by conditioning. For a single sample path $X^H[0, 1](\omega) = \{X^H_t(\omega) : 0 \leq t \leq 1\}$, one knows its Hausdorff dimension (as a subset of $\mathbb{R}^n$) explicitly in terms of H (see Theorem 3.2). On the other hand, it is also known that for any bounded Borel set $E \subset \mathbb{R}^d$ the probability

$$\mathbb{P}\,(X_t \text{ hits } E \text{ for } t \in [a, b])$$

can be characterized by the α-dimensional Newtonian capacity of E for $\alpha = n - 1/K$ (see Theorem 3.1). Given the relation between Hausdorff dimension and Capacity dimension, one should be able to draw some information on whether X^K hits a particular sample path $E = X^H[0, 1](\omega)$ of X^H. The question whether X^K hits X^H is then answered by undoing the conditioning.

Throughout our discussion, we assume that the vector fields V_i (and $\tilde{V}_i$, respectively) in Eq. (1) for X^H (and X^K, respectively) are C^∞-bounded and satisfy the following uniform ellipticity condition.

Hypothesis 1.1 (*Uniform Ellipticity*)**.** The vector fields $V_1, \ldots, V_d$ are said to form an uniform elliptic system if

$$v^* V(x) V^*(x) v \geq \lambda |v|^2, \qquad \text{for all } v, x \in \mathbb{R}^n, \tag{2}$$

where we have set $V = (V^i_j)_{i=1,\ldots,n; j=1,\ldots,d}$ and where λ designates a strictly positive constant.

Remark 1.2. Under the uniform ellipticity condition we have $d \geq n$.

The main result of our investigation is reported in the following two theorems.

Theorem 1.3. *Consider the event*

$$A = \{X^H \text{ and } X^K \text{ intersect each other over the interval } [0, 1]\}.$$

We have

(1) *if* $n > 1/H + 1/K$, *then* $\mathbb{P}(A) = 0$*; and*
(2) *if* $n < 1/H + 1/K$, *then* $\mathbb{P}(A) > 0$.

Fix any small $\epsilon > 0$, and let $T = \{(s, t) \in [\epsilon, 1]^2 : X^H_s = X^K_t\}$. Denote by $\dim_{\mathcal{H}} T$ and $\dim_{\mathcal{P}} T$ the Hausdorff dimension and packing dimension of T, respectively.

Theorem 1.4. *If* $\frac{1}{H} + \frac{1}{K} > n$, *then with positive probability,*

$$\dim_{\mathcal{H}} T = \dim_{\mathcal{P}} T = \begin{cases} 2 - nH & \text{if} \quad \dfrac{1}{H} > n, \\[2ex] 1 + \dfrac{K}{H} - nK & \text{if} \quad \dfrac{1}{H} \leq n < \dfrac{1}{H} + \dfrac{1}{K}. \end{cases}$$

The rest of the paper is organized as follows. In Section 2, we present some preliminary material on rough path theory and stochastic differential equations driven by fractional Brownian motions. The needed results on fractal properties of solutions to Eq. (1) is summarized in Section 3. Finally, we prove Theorem 1.3 in Section 4, and Theorem 1.4 in Section 5.