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Feasible blocked multi-factor designs of unequal block sizes

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ABSTRACT

A block design is called feasible if all the factor effects in it are estimable. This paper studies the design structures of feasible block designs when the block sizes are unequal. Based on a necessary and sufficient condition, we find two structure features which respectively lead to unfeasible and feasible designs. According to these results, an effective way to find feasible designs is provided.

1. Introduction

In design of experiments, to reduce the variability of a noise source that is not of primary interest to the experimenter, blocking is to arrange the experimental units in groups (blocks) that are similar to one another. How to settle the factor's levels in different blocks motivates the issue of selecting good block designs.

There is a large body of literature on the construction of block designs. Many of the results deal with one factor problem. Related studies include balanced incomplete block design (BIBD), pairwise balanced design (PBD) and partially balanced incomplete block design (PBIBD), see Raghavarao and Padgett (2005) for reference.

Recently, construction of block design that arranges multiple factors is studied. Sitter et al. (1997), Chen and Cheng (1999), Cheng and Wu (2002) and Dey (2010) gave minimum-aberration criterions for blocked fractional factorial designs. Then the construction of blocked optimal designs is proposed, including regular orthogonal block designs (Cheng and Tsai, 2009) and non-regular blocked two-level designs (Cheng et al., 2004; Das and Dey, 2004). These results deal primarily with orthogonal designs. Several results are also available for the construction of blocked non-orthogonal designs, such as Bagchi (2010), Jacroux (2013) and Chen et al. (2015). However, all such results given are under the assumption that all blocks have equal sizes, with the constrain of two-level or three-level factors.

When designing a blocked multi-factor design, one should consider the situation that the blocks have unequal sizes for practical limitation. This paper considers experimental situations in which m factors are to be studied in n runs which are partitioned into b blocks of unequal sizes and where only the main effects of the factors are to be used. A good introduction to such types of models can be seen in John and Williams (1995). The number of the levels of each factor is set to arbitrary integer larger than 1. This complicates the issue of selecting a proper design. In this study, we aim to exploring the structure of feasible block designs, wherein all the factors' main effects are estimable. By imposing a projection matrix B_t, we obtain a necessary and sufficient condition for feasibility of a block design. Then both classes of designs which are unfeasible and feasible are found. Based on the results, an effective way for finding feasible designs is provided. In Section 2, we give our basic notations and definitions. In Section 3, we propose our main results concerning feasibility and the algorithms for selecting feasible block designs are given in Section 4. Conclusions are drawn in Section 5.

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2. Basic notation and definitions

Throughout this paper, we use H to denote a block design having m factors occur in n runs arranged in b blocks of sizes $k_1, \ldots, k_b$. We shall represent H by $H = (\boldsymbol{h_0}; \boldsymbol{h_1}, \ldots, \boldsymbol{h_m}) = (h_{pt})$, $p = 1, \ldots, n$, $t = 0, 1, \ldots, m$, which is an $n \times (m+1)$ matrix whose pth row corresponds to run p, the first column indicates the blocks equaling $(\mathbf{1}'_{\boldsymbol{k_1}}, 2\mathbf{1}'_{\boldsymbol{k_2}}, \ldots, b\mathbf{1}'_{\boldsymbol{k_b}})'$, where $\mathbf{1}_{\boldsymbol{k_i}}$ is the $k_i \times 1$ vector of ones, and $(t+1)$th column $(t \geq 1)$ corresponds to factor t whose entries $h_{pt} \in \{1, \ldots, v_t\}$, where v_t is the number of levels of factor t.

The model to be used for analyzing the data under a given block design H is

$$\boldsymbol{Y} = C_0\boldsymbol{\alpha} + C_1\boldsymbol{\beta_1} + \cdots + C_m\boldsymbol{\beta_m} + \boldsymbol{\varepsilon}, \tag{1}$$

where $\boldsymbol{Y}$ is an $n \times 1$ vector of observations, C_0 is the block incidence matrix which is an $n \times b$ matrix with (p, q)th element being 1 or 0, $p = 1, \ldots, n$, $q = 1, \ldots, b$, depending on whether the pth run occurs in the qth block, $\boldsymbol{\alpha}' = (\alpha_1, \ldots, \alpha_b)$ is the vector of block parameters, C_t is the incidence matrix (Zhang et al., 1999; Raghavarao and Padgett, 2005; Chen et al., 2015) of factor t, $t = 1, \ldots, m$, which is an $n \times v_t$ matrix with (p, q)th element being 1 or 0, $p = 1, \ldots, n$, $q = 1, \ldots, v_t$, depending on whether $h_{pt} = q$, $\boldsymbol{\beta}'_{\boldsymbol{t}} = (\beta_{t1}, \ldots, \beta_{tv_t})$ is the vector of main effects of factor t which has a constraint

$$\beta_{t1} + \cdots + \beta_{tv_t} = 0, \tag{2}$$

and $\boldsymbol{\varepsilon}$ is a vector of random error terms whose entries are assumed to be uncorrelated with the zero mean and constant variance σ^2.

According to Eq. (2),

$$\boldsymbol{\beta_t} = \begin{pmatrix} I_{v_t-1} \\ -\mathbf{1}'_{\boldsymbol{v_t-1}} \end{pmatrix} \boldsymbol{\beta}^*_{\boldsymbol{t}}$$

where $\boldsymbol{\beta}^*_{\boldsymbol{t}} = (\beta_{t1}, \ldots, \beta_{tv_t-1})'$. Thus we shall interchangeably represent the model as

$$\boldsymbol{Y} = C_0\boldsymbol{\alpha} + D_1\boldsymbol{\beta}^*_{\boldsymbol{1}} + \cdots + D_m\boldsymbol{\beta}^*_{\boldsymbol{m}} + \boldsymbol{\varepsilon}, \tag{3}$$

where $D_t = C_t \begin{pmatrix} I_{v_t-1} \\ -\mathbf{1}'_{\boldsymbol{v_t-1}} \end{pmatrix}$.

Definition 2.1. A block design H is called feasible if all the factor effects are estimable. That is, for each $\boldsymbol{\beta_t}$, there is a matrix F_t such that

$$E(F_tY) = \boldsymbol{\beta_t}.$$

Let $r_i^t(x)$ denote the frequency that level x of factor t occurs in the ith block. Then the meeting times of arbitrary two different levels x and y in block i should be $r_i^t(x)r_i^t(y)$.

Definition 2.2. Denote $\lambda^t(x, y) = \sum_{i=1}^{b} \frac{r_i^t(x)r_i^t(y)}{k_i}$ as the meeting degree of levels x and y for factor t. Then H is called meeting balanced if for $t = 1, \ldots, m$,

$$\lambda^t(x, y) = \lambda_t \tag{4}$$

for arbitrary x and y, where λ_t is a constant not 0.

3. Feasibility results

In this section, we give our feasible results for H under the meeting balanced condition. We first establish a necessary and sufficient condition for feasibility by some projection matrices B_t, and then obtain the feasible design structures from the properties of B_t.

3.1. Preliminary

First we give and prove three lemmas.

Lemma 3.1. *Let* $P_{C_0} = C_0(C_0'C_0)^-C_0'$, $\tau = I_n - P_{C_0}$, $P_{v_t} = \frac{1}{v_t}\mathbf{1}_{\boldsymbol{v_t}}\mathbf{1}'_{\boldsymbol{v_t}}$ *and* $\tau_{v_t} = I_{v_t} - P_{v_t}$. *Then* Eq. (4) *is equivalent to*

$$C_t'\tau C_t = \lambda_t v_t \tau_{v_t} \tag{5}$$

for $t = 1, \ldots, m$.

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