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Generalized skew-elliptical distributions are closed under affine transformations

Tomer Shushi *

Faculty of Natural Sciences, Department of Physics, Ben-Gurion University of the Negev, Beersheba 8410501, Israel
Actuarial Research Center, Department of Statistics, University of Haifa, Mount Carmel, Haifa 3498838, Israel

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In this short letter we prove that the family of multivariate generalized skew-elliptical distributions is closed under affine transformations. This fundamental property has many applications in applied and theoretical probability.

1. Introduction

One of the main important properties of the multivariate normal distribution is that it is closed under affine transformations, which means that for an $m \times n$ matrix B of rank m, $m < n$, the random vector $B\mathbf{X}$, $\mathbf{X} \backsim N_n(\mu, \Sigma)$, is also a normal random vector, $B\mathbf{X} \backsim N_m(B\mu, B\Sigma B^T)$. Furthermore, this property also holds for every member of the elliptical family of distributions

$$f_{\mathbf{X}}(\mathbf{x}) = |\Sigma|^{-1/2} g_n\left(\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma^{-1}(\mathbf{x}-\mu)\right), \mathbf{x} \in \mathbb{R}^n, \tag{1}$$

where $\mathbf{X} \backsim E_n(\mu, \Sigma, g_n)$ is the elliptical random vector with the density generator $g_n(u) \geq 0$, μ is $n \times 1$ vector of means, Σ is an $n \times n$ positive definite matrix.

A well-known generalization of the elliptical family, into the world of skewed distributions, is the generalized skew-elliptical (GSE) family of distributions which takes the following form (Genton and Loperfido, 2005)

$$f_{\mathbf{Y}}(\mathbf{y}) = 2|\Sigma|^{-1/2} g_n\left(\frac{1}{2}(\mathbf{y}-\mu)^T \Sigma^{-1}(\mathbf{y}-\mu)\right) \pi(\Sigma^{-1/2}(\mathbf{y}-\mu)), \mathbf{y} \in \mathbb{R}^n, \tag{2}$$

where π is called the *skewing function* which is defined as a mapping from the space of random vectors $\mathbb{R}^n$ to the real (non-negative) line $\mathbb{R}_+$, i.e.,

$$\pi : \mathbb{R}^n \ni \mathbf{Y} \Longrightarrow \pi(\mathbf{Y}) \geq 0.$$

Furthermore, π satisfies the following properties

$$0 \leq \pi(\mathbf{y}) \leq 1, \forall \mathbf{y} \in \mathbb{R}^n$$

* Correspondence to: Faculty of Natural Sciences, Department of Physics, Ben-Gurion University of the Negev, Beersheba 8410501, Israel.
E-mail address: tomershu@post.bgu.ac.il.

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and

$$\pi(-\mathbf{y}) = 1 - \pi(\mathbf{y}), \forall \mathbf{y} \in \mathbb{R}^n.$$

We say that an $n \times 1$ random vector $\mathbf{Y}$ is a GSE random vector if its probability density function (pdf) takes the form of (2), and we write $\mathbf{Y} \backsim GSE_n(\mu, \Sigma, g_n, \pi)$.

The characteristic function of special members of the GSE family was computed in the literature (see, for instance, Azzalini and Valle, 1996; Loperfido, 2004; Nadarajah and Kotz, 2003; Vernic, 2005, and Shushi, 2016, 2017). In Shushi (2016), it was proved that the characteristic function of any GSE random vector $\mathbf{Y}$ takes the following well-defined form

$$c(\mathbf{t}) = 2e^{i\mathbf{t}^T\mu}\Psi_n\left(\frac{1}{2}\mathbf{t}^T\Sigma\mathbf{t}\right)k_n(\mathbf{t}), \mathbf{t} \in \mathbb{R}^n,$$

where Ψ_n is the characteristic function of the $n \times 1$ elliptical random vector $\mathbf{X}$, and k_n is a function that satisfies the following properties

$$k_n(-\mathbf{t}) = 1 - k_n(\mathbf{t}),$$

and

$$0 \le k_n(\mathbf{t}) \le 1.$$

For the sequel, we define a quasi-pdf of a random vector $\mathbf{Z}^{\mathbf{t}}_{n,\Sigma}$,

$$p_{\mathbf{t}}(\mathbf{z}) = \exp\left(i\mathbf{t}^T\mathbf{z}\right)g_n\left(\frac{1}{2}\mathbf{z}^T\Sigma^{-1}\mathbf{z}\right)/\Psi_n\left(\frac{1}{2}\mathbf{t}^T\Sigma\mathbf{t}\right), \mathbf{z} \in \mathbb{R}^n.$$

We now prove that the GSE family of distributions is closed under affine transformations.

Theorem 1. *Let $\mathbf{Y} \backsim GSE_n(\mu, \Sigma, g_n, \pi)$. Then, for any $\mathbf{a} \in \mathbb{R}^m$, and $m \times n$ matrix B of rank m, $m < n$, the affine transformation $\mathbf{a} + B\mathbf{Y}$ is also a GSE random vector with the characteristic function*

$$c_{B\mathbf{Y}}(\mathbf{t}) = 2e^{i\mathbf{t}^T(\mathbf{a}+B\mu)}\Psi_m\left(\frac{1}{2}\mathbf{t}^TB\Sigma B^T\mathbf{t}\right)k_m(\mathbf{t}), \mathbf{t} \in \mathbb{R}^m.$$

Here

$$k_m(\mathbf{t}) = E\left(\widetilde{\pi}\left(\left(B\Sigma B^T\right)^{-1/2}\mathbf{Z}^{\mathbf{t}}_{m,B\Sigma B^T}\right)\right),$$

where $\widetilde{\pi}(\mathbf{u}) = \pi\left(\Sigma^{-1/2}B^\dagger\left(B\Sigma B^T\right)^{1/2}\mathbf{u}\right)$, $\mathbf{u} \in \mathbb{R}^m$, and $B^\dagger$ is the Moore–Penrose pseudoinverse of B from the left side.

Proof. From the definition of the characteristic function of the GSE distributions, and after taking the transformation $\mathbf{v} = \Sigma^{-1/2}(\mathbf{y} - \mu)$, we have

$$\begin{aligned} c_{B\mathbf{Y}}(\mathbf{t}) &= 2|\Sigma|^{-1/2}e^{i\mathbf{t}^T\mathbf{a}}\int_{\mathbb{R}^n} e^{i\left(B^T\mathbf{t}\right)^T\mathbf{y}}g_n\left(\frac{1}{2}(\mathbf{y}-\mu)^T\Sigma^{-1}(\mathbf{y}-\mu)\right)\pi\left(\Sigma^{-1/2}(\mathbf{y}-\mu)\right)d\mathbf{y} \\ &= 2e^{i\mathbf{t}^T(\mathbf{a}+B\mu)}\int_{\mathbb{R}^n} e^{i\mathbf{t}^T\left(B\Sigma^{1/2}\mathbf{v}\right)}g_n\left(\frac{1}{2}\mathbf{v}^T\mathbf{v}\right)\pi(\mathbf{v})\,d\mathbf{v}, \mathbf{t} \in \mathbb{R}^m. \end{aligned}$$

Taking the Moore–Penrose pseudoinverse of $A = B\Sigma^{1/2}$ from the left side, $A^\dagger$, and taking into account the existence and uniqueness of $A^\dagger$, we define a new skewing function π^* such that

$$\pi^*(A\mathbf{v}) = \pi\left(A^\dagger A\mathbf{v}\right),$$

where $A^\dagger = \left(B\Sigma^{1/2}\right)^\dagger = \Sigma^{-1/2}B^\dagger$, which is given from the properties of the pseudoinverse matrix.

Thus, we conclude that

$$c_{B\mathbf{Y}}(\mathbf{t}) = 2e^{i\mathbf{t}^T(\mathbf{a}+B\mu)}\int_{\mathbb{R}^n} e^{i\mathbf{t}^T(A\mathbf{v})}g_n\left(\frac{1}{2}\mathbf{v}^T\mathbf{v}\right)\pi^*(A\mathbf{v})\,d\mathbf{v} = E\left(\Lambda(A\mathbf{X})\right), \quad (3)$$

where

$$\Lambda(A\mathbf{X}) = e^{i\mathbf{t}^T(A\mathbf{X})}\pi^*(A\mathbf{X}),$$

and $E_{\mathbf{X}}(\Lambda(A\mathbf{X}))$ is the expected value of $\Lambda(A\mathbf{X})$ respect to the elliptical distribution, $\mathbf{X} \backsim E_n(\mathbf{0}, I_n, g_n)$, where I_n is the $n \times n$ identity matrix. Then, from the marginal properties of the elliptical distributions (see, Fang et al., 1990), it is clear that

$$A\mathbf{X} \backsim E_m(\mathbf{0}, AA^T, g_m).$$