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Excessive measures for linear diffusions

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ABSTRACT

Excessive measures play an important role in the study of Markov processes. However, there are few works about the existence of excessive measures. In this paper, we give a necessary and sufficient condition for a linear diffusion to possess a fully supported excessive measure.

1. Introduction and main result

Excessive measures and related potential theory associated with a right Markov process have been studied intensively since the pioneering work by Fitzsimmons and Maisonneuve (Fitzsimmons and Maisonneuve, 1986). An interesting thing is that any excessive measure can be used to construct a certain measure on the path space, which is called Kuznetsov measure. By a series works of Fitzsimmons and Maisonneuve, Kuznetsov measure has been proved to be an essential tool to develop probabilistic potential theory. Following these works there was remarkable progress in the study of excessive measures, and it had received more attention than excessive functions which used to be the main role in potential theory as shown in Blumenthal and Getoor (1968). The book (Getoor, 1990), published in 1990 by R. Getoor, summarized these classical results concerning excessive measures. However when we look back, we find that there is one problem we have not answered until now, that is, the existence of excessive measures.

It is known that the existence of excessive functions is trivial, because any constant is always an excessive function. But the existence of excessive measures is far from trivial except in some special cases. Though the existence and uniqueness of invariant measures for Markov processes have been attracting problems, the existence of fully supported or even non-trivial excessive measures has never been seriously discussed.

In this paper, we give a necessary and sufficient condition for a linear diffusion to possess a fully supported excessive measure. A linear (or 1-dimensional) diffusion is a continuous stochastic process with the strong Markov property on an interval of real line, which has been widely studied. Many classical results can be found in books such as Itô (2006), Itô and McKean (1974), and Rogers and Williams (2000). Under the 'regular' condition, a linear diffusion can be characterized by a scale function, a speed measure and a killing measure. It is also possible to decompose a general linear diffusion into non-communicating pieces (see Itô and McKean, 1974, Section 3.5). This idea will be useful in this paper.

Turning to the context of Markov processes, let (P_t) be a transition semigroup of kernels on a measurable space $(E, \mathcal{E})$. A σ-finite measure ξ on E is called **excessive** if $\xi P_t \leq \xi,\ \forall t > 0$, and **invariant** if $\xi P_t = \xi,\ \forall t > 0$, where

$$\xi P_t := \int_E \xi(dx) P_t(x, \cdot).$$

For any $\alpha \geq 0$, the potential operator U^α is defined to be

$$U^\alpha(x, A) = \int_0^\infty e^{-\alpha t} P_t(x, A) dt = \mathbf{E}_x \int_0^\infty e^{-\alpha t} 1_{\{X_t \in A\}} dt.$$

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We write U for U^0. For any measure μ on E, one can easily check that $(\mu U)P_t \leq \mu U$ and so μU is an excessive measure as long as it is σ-finite. An excessive measure of form μU is called a potential, which plays an essential role in this paper.

By saying a diffusion $X = (X_t)$ on interval I, we mean that the state space of X is $(I, \mathscr{B}(I))$. Thus an excessive measure ξ of X is defined on I, and it is fully supported if $\xi(J) > 0$ for any non-empty open interval $J \subset I$. A cemetery point Δ is adjoined to I as the one-point compactification and $\zeta = \inf\{t > 0 : X_t = \Delta\}$ is the lifetime of X. We always assume in this paper that the continuity of X_t holds on $[0, \infty)$, which means that there is no killing inside I and X can only enter the cemetery point from an open boundary.

We call a set $B \subset I$ **recurrent** if

(1) $\mathbf{P}_x(X_t \in B, \forall t) = 1$ for all $x \in B$,
(2) $\mathbf{P}_x(\sigma_y < \infty) = 1$ for all $x, y \in B$ (x may equal to y),

where $\sigma_y := \inf\{t > 0 : X_t = y\}$. Our main result is the following theorem.

Theorem 1.1. *Let X be a diffusion on interval I. Then X possesses a fully supported excessive measure if and only if $\mathbf{P}_x(\sigma_B < \infty) = 0$ for any recurrent set B and $x \in I \setminus \overline{B}$.*

Remark 1.2. The set of excessive measures depends on the choice of state space. For example, let $X_t = t$ on $[0, 1]$ with 1 as an absorbing barrier. If the state space is chosen to be $[0, 1]$, then $\{1\}$ is a recurrent set which is accessible, so that there are no fully supported excessive measures. If the state space may be chosen to be $[0, 1)$ and $\{1\}$ is cemetery point, then there is no recurrent set so that there exist fully supported excessive measures. In fact, the Lebesgue measure on $[0, 1)$ is excessive.

2. Proof of main result

To prove our main result, we need to prepare some results for regular diffusions. Most of them are known or expected but may not be written explicitly.

A diffusion $X = (X_t)$ on interval I is called **regular** if for all $x \in \mathring{I},\ y \in I$,

$$\mathbf{P}_x(\sigma_y < \infty) > 0,$$

where $\mathring{I}$ is the interior of I. An endpoint c of the interval I is called

(1) inaccessible if $c \notin I$;
(2) absorbing if $c \in I$ and $\mathbf{P}_c(\sigma_y < \infty) = 0$ for all $y \in I \setminus \{c\}$;
(3) reflecting if $c \in I$ and $\mathbf{P}_c(\sigma_y < \infty) > 0$ for some $y \in I \setminus \{c\}$.

We regard absorbing endpoints as cemetery point, and the state space becomes

$$I' = I \setminus \{\text{absorbing endpoints}\}.$$

Then X is irreducible, i.e. every two states are accessible. X is called **conservative** if

$$\mathbf{P}_x(\zeta = \infty) = 1$$

for any $x \in I'$. Note that a regular diffusion is symmetric with respect to its speed measure m. Hence m is excessive, and invariant if X is conservative.

The irreducibility leads to the property that X is either recurrent or transient. There are many different forms of definition for recurrence and transience, and for regular diffusion, they are all equivalent. In this paper, we call a regular diffusion on I **recurrent** if

$$\mathbf{P}_x(\sigma_y < \infty) = 1, \quad \forall x, y \in I'. \tag{2.1}$$

This definition is equivalent to (iv) of Theorem 1 in Chung and Walsh (2005), Section 3.7 for a regular diffusion. The theorem also says that recurrence implies conservativeness. Proposition 2.4 in Getoor (1980) gives several other equivalent definitions from which we pick one for later use:

$$U(\,\cdot\,, B) \equiv 0 \text{ or } \infty, \quad \forall B \subset I'. \tag{2.2}$$

Let

$$\sigma_y^r := \inf\{t > r : X_t = y\} = r + \sigma_y \circ \theta_r.$$

We call a regular diffusion **transient** if

$$\lim_{r\to\infty} \mathbf{P}_x(\sigma_y^r < \infty) = 0, \quad \forall x, y \in I'. \tag{2.3}$$