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Triangular random matrices and biorthogonal ensembles

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ABSTRACT

We study the singular values of certain triangular random matrices. When their elements are i.i.d. standard complex Gaussian random variables, the squares of the singular values form a biorthogonal ensemble, and with an appropriate change in the distribution of the diagonal elements, they give the biorthogonal Laguerre ensemble.

1. Introduction and statement of the results

1.1. Singular values of random matrices

Singular values of random matrices are of importance in numerical analysis, multivariate statistics, information theory, and the spectral theory of random non-symmetric matrices. See the survey paper (Chafaï, 2009). The starting point in this field is the result of Marchenko and Pastur (1967) (see also Theorem 3.6 in Bai and Silverstein, 2010 for a more recent exposition), which is the following.

Let $\{X_{i,j} : i, j \in \mathbb{N}^+\}$ be i.i.d. complex valued random variables with variance 1, and for $n, m \in \mathbb{N}^+$ consider the $n \times m$ matrix $X(n, m) := (X_{i,j})_{1\le i\le n, 1\le j\le m}$. Call $\lambda_1^{n,m} \ge \lambda_2^{n,m} \ge \cdots \ge \lambda_n^{n,m} \ge 0$ the eigenvalues of the Hermitian, positive definite matrix

$$S_{n,m} = \frac{1}{m} X(n, m)X(n, m)^*,$$

and

$$L_{n,m} := \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i^{n,m}}$$

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their empirical distribution. Then for every $c > 0$, with probability 1, as $n, m \to \infty$ so that $n/m \to c$, $L_{n,m}$ converges weakly to the measure

$$\mathbf{1}_{a\leq x\leq b}\frac{1}{2\pi xc}\sqrt{(b-x)(x-a)}\,dx+\mathbf{1}_{c>1}\left(1-\frac{1}{c}\right)\delta_0 \tag{1}$$

where $a = (1-\sqrt{c})^2,\ b = (1+\sqrt{c})^2$.

This is a universality result as the limit does not depend on the fine details of the distribution of the matrix elements $X_{i,j}$. On the other hand, the joint distribution of the eigenvalues $(\lambda_1^{n,m}, \lambda_2^{n,m}, \ldots, \lambda_n^{n,m})$, not surprisingly, depends on the exact distribution of the matrix elements. In a few cases this joint distribution can be determined. For example, if the $X_{i,j}$ follow the standard complex Gaussian distribution and $n \leq m$, the vector $(\lambda_1^{n,m}, \lambda_2^{n,m}, \ldots, \lambda_n^{n,m})$ has density with respect to Lebesgue measure in $\mathbb{R}^n$ which is

$$\frac{1}{\prod_{k=1}^n \Gamma(m-n+k)\Gamma(k)}e^{-\sum_{k=1}^n x_k}\left(\prod_{k=1}^n x_i\right)^{m-n}\prod_{1\leq i\leq j\leq n}(x_i-x_j)^2\mathbf{1}_{x_1>x_2>\cdots>x_n>0}. \tag{2}$$

See, for example, relation (3.16) in Forrester (2010).

1.2. Triangular Gaussian matrices

In this work, we study the singular values of certain triangular random matrices. The motivation comes from the purely mathematical viewpoint as triangular matrices are ingredients in several matrix decompositions.

Assume as above that $\{X_{i,j} : i, j \in \mathbb{N}^+, i \geq j, \}$ are i.i.d. complex valued random variables with variance 1, and for $n \in \mathbb{N}^+$ let $X(n)$ be the lower triangular $n \times n$ matrix whose (i, j) element is $X_{i,j}$ for $1 \leq j \leq i \leq n$. Call $\lambda_1^{(n)} \geq \lambda_2^{(n)} \geq \cdots \geq \lambda_n^{(n)} \geq 0$ the eigenvalues of the Hermitian matrix

$$S_n = \frac{1}{n}X(n)X(n)^*,$$

and

$$L_n := \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i^{(n)}}$$

their empirical distribution.

The fact that L_n converges weakly and description of the limit were given in Dykema and Haagerup (2004). It is analogous to the result of Marchenco and Pastur mentioned in the previous section and it says that with probability 1 the sequence $(L_n)_{n\geq 1}$ converges weakly to a deterministic measure μ_0 on $\mathbb{R}$ with moments

$$\int_{\mathbb{R}} x^k d\mu_0(x) = \frac{k^k}{(k+1)!} \tag{3}$$

for all $k \in \mathbb{N}$. The measure μ_0 is absolutely continuous with density that has support $[0, e]$ and can be expressed in terms of the Lambert function.

Here we do the obvious next step. That is, explore cases of distributions for the elements of the matrix $X(n)$ for which the joint distribution of the eigenvalues of $X(n)X(n)^*$ can be computed. The first such case is the following.

Theorem 1. *Let $n \in \mathbb{N}^+$ and assume that the random variables $\{X_{i,j} : i, j \in \mathbb{N}^+,\ i \geq j\}$ are complex standard normal. Then:*

(i) *The vector $\Lambda_n := (\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_n^{(n)})$ of the eigenvalues $\lambda_1^{(n)} \geq \lambda_2^{(n)} \geq \cdots \geq \lambda_n^{(n)}$ of $X(n)X^*(n)$ has density given by*

$$f_{\Lambda_n}(x_1, x_2, \ldots, x_n) = \frac{1}{\prod_{j=1}^{n-1} j!}e^{-\sum_{j=1}^n x_j}\prod_{i<j}(x_i-x_j)(\log x_i - \log x_j)\mathbf{1}_{x_1>x_2>\cdots>x_n>0}. \tag{4}$$

(ii) *The point process $\{\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_n^{(n)}\}$ is determinantal.*

The theorem will be implied by the more general Theorems 2 and 3 of the next subsection.

1.3. Eigenvalue realization of the biorthogonal Laguerre ensemble

The next model of random triangular matrix that we study is one where the elements strictly below the diagonal are i.i.d. standard complex normal but the elements of the diagonal are independent but not identically distributed.

More specifically, fix a positive integer n, reals $\theta \geq 0$, $b > 0$, and let

$$c_k = \theta(k-1) + b \tag{5}$$