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Multidimensional extremal dependence coefficients

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ABSTRACT

Extreme value modeling has been attracting the attention of researchers in diverse areas such as the environment, engineering, and finance. Multivariate extreme value distributions are particularly suitable to model the tails of multidimensional phenomena. The analysis of the dependence among multivariate maxima is useful to evaluate risk. Here we present new multivariate extreme value models, as well as, coefficients to assess multivariate extremal dependence.

1. Introduction

Let $\mathbf{X} = \{X(\mathbf{x}), \mathbf{x} \in \mathbb{R}^m\}$ be a random field, $I = \{1, \ldots, d\}$ and consider $I_1 = \{1, \ldots, i_1\}$, $I_2 = \{i_1 + 1, \ldots, i_2\}$, ..., $I_p = \{i_{p-1} + 1, \ldots, i_p = d\}$ a partition of I, $1 \le p \le d$. For a fixed set of locations $L = \{\mathbf{x}_j : j \in I\} \subset \mathbb{R}^m$ and some partition $L_j = \{\mathbf{x}_i : i \in I_j\}$, $j = 1, \ldots, p$, with $1 \le p \le d$, consider the random vectors $\mathbf{X}_{I_1} = (X(\mathbf{x}_1), \ldots, X(\mathbf{x}_{i_1}))$, ..., $\mathbf{X}_{I_p} = (X(\mathbf{x}_{i_{p-1}+1}), \ldots, X(\mathbf{x}_d))$. We are going to evaluate the dependence between the vectors through coefficients, that is, the dependence between the marginals of $\mathbf{X}$ over disjoint regions $L_1, \ldots, L_p$.

Examples of applications within this context can be found in Naveau et al. (2009) and Guillou et al. (2014) for $d = p = 2$, i.e., two locations, in Fonseca et al. (2015) for $d > 2$ and $p = 2$, i.e., two groups of several locations and Ferreira and Pereira (2015) for $d = p > 2$, i.e., several isolated locations. More precisely, in Naveau et al. (2009) was inferred the dependence between maxima of daily precipitation in pairwise locations of Bourgogne (Dijon), Guillou et al. (2014) address the dependence between the monthly maxima of hourly precipitation of two stations from a hydrological basin in Orgeval (Paris), in Fonseca et al. (2015) is assessed the dependence between annual maxima values of daily maxima rainfall in several regions of Portugal and Ferreira and Pereira (2015) evaluate the dependence within the annual maxima of tritium (pCi/L) in drinking water for three locations in Alabama State (USA).

In the applications, in order to study the dependence between sub-vectors of $\mathbf{X}$ we can form an auxiliary vector $(Y_1, \ldots, Y_p)$ where each variable Y_j somehow summarizes the information of $\mathbf{X}_{I_j}$, $j = 1, \ldots, p$, and study the dependence between the variables Y_j. This is the approach followed by some authors (Naveau et al., 2009; Marcon et al., in press). In our proposal to infer the dependence between clusters of variables, we deal directly with the vectors $\mathbf{X}_{I_j}$, $j = 1, \ldots, p$. On the

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other hand, if the random field is vectorial, that is, for each location $\mathbf{x}_i$, $X(\mathbf{x}_i)$ is a vector $(X^1(\mathbf{x}_i), \ldots, X^s(\mathbf{x}_i))$, whenever we think of the dependence between $X(\mathbf{x}_1), \ldots, X(\mathbf{x}_d)$ we have dependency between vectors.

The dependence between the random vectors $\mathbf{X}_{I_1}, \mathbf{X}_{I_2}, \ldots, \mathbf{X}_{I_p}$ can be characterized through the exponent measure

$$\ell_{\mathbf{x}_1,\ldots,\mathbf{x}_d}(t_1, \ldots, t_d) = -\ln F_{(X(\mathbf{x}_1),\ldots,X(\mathbf{x}_d))}(t_1, \ldots, t_d),$$

where $F_{(X(\mathbf{x}_1),\ldots,X(\mathbf{x}_d))}$ denotes the distribution function (df) of $\mathbf{X}_I = (X(\mathbf{x}_1), \ldots, X(\mathbf{x}_d))$. If $\mathbf{X}$ is a max-stable random field with unit Fréchet marginals, then $\ell_{\mathbf{x}_1,\ldots,\mathbf{x}_d}$ is homogeneous of order -1 and the polar transformation used in the Pickands representation allows us to see it as a moment-based tail dependence tool (see, e.g., Finkenstädt and Rootzén, 2003 or Beirlant et al., 2004).

Our proposal also addresses $\ell_{\mathbf{x}_1,\ldots,\mathbf{x}_d}$ as a function of moments of transformations of $\mathbf{X}_I$. Specifically, the moments

$$e(\lambda_1, \ldots, \lambda_p) = E\left(\bigvee_{j=1}^{p}\bigvee_{i\in I_j} F^{\lambda_j}_{X(\mathbf{x}_i)}(X(\mathbf{x}_i))\right), \ (\lambda_1, \ldots, \lambda_p) \in (0, \infty)^p,$$

where $a \vee b = \max(a, b)$. If $p = d = 2$, $\frac{1}{2}e(\lambda, 1-\lambda)$ equals the λ-madogram of Naveau et al. (2009), unless the addition of constant $\frac{1}{2}(E(U^\lambda + E(U^{1-\lambda}))$ where U is standard uniform. When $p = d \geq 2$, $e(\lambda_1^{-1}, \ldots, \lambda_d^{-1})$ with $\sum_{j=1}^d \lambda_j = 1$ equals the generalized madogram considered in Marcon et al. (in press), unless the addition of constant $\frac{1}{d}\sum_{j=1}^d E\left(U^{\lambda_j^{-1}}\right)$.

Here we also consider a shifted $e(\lambda_1, \ldots, \lambda_p)$ by subtracting the constant

$$\frac{1}{p}\sum_{i=1}^{p} E\left(\bigvee_{i\in I_j} F^{\lambda_j}_{X(\mathbf{x}_i)}(X(\mathbf{x}_i))\right).$$

The referred works consider max-stable random fields with standard Fréchet marginals, except Guillou et al. (2014) where $\ell_{x_1,x_2}(t_1, t_2)$ is homogeneous of order $-1/\eta$ and $F_{X(x_i)}(t) = P(X(x_i) \leq t) = \exp(-\sigma(x_i)t^{-1/\eta}$, $i = 1, 2$, $\eta \in (0, 1]$, corresponding to the bivariate extreme values model obtained in Ramos and Ledford (2011).

We will also consider that $F_{(X(\mathbf{x}_1),\ldots,X(\mathbf{x}_d))}$ is such that $\ell_{\mathbf{x}_1,\ldots,\mathbf{x}_d}(t_1, \ldots, t_d)$ is homogeneous of order $-1/\eta$ and $F_{X(\mathbf{x}_j)}(t) = P(X(\mathbf{x}_j) \leq t) = \exp(-\sigma(\mathbf{x}_j)t^{-1/\eta})$, $j = 1, \ldots, d$, for some constants $\sigma(\mathbf{x}_j) > 0$ and $\eta \in (0, 1]$. Under this hypothesis, which includes all the other mentioned works whenever $\eta = 1$ and $\sigma(\mathbf{x}_j) = 1$, we define extremal dependence functions that provide us coefficients to measure the dependence among $\mathbf{X}_{I_1}, \ldots, \mathbf{X}_{I_p}$ through the dependence between $M(I_j) = \bigvee_{i\in I_j} F_{X(\mathbf{x}_i)}(X(\mathbf{x}_i))$, $j = 1, \ldots, p$. We relate the extremal coefficients with the upper tail dependence function introduced in Ferreira and Ferreira (2012), which was extended to random fields in Pereira et al. (2017). This is addressed in Section 2. We compute the extremal coefficients for several choices of $F_{(X(\mathbf{x}_1),\ldots,X(\mathbf{x}_d))}$ in Section 3. Finally we consider an asymptotic tail independence coefficient to measure an "almost" independence for a class of models wider than max-stable ones (Section 4).

In order to simplify notations, we will write X_i instead of $X(\mathbf{x}_i)$ and, for any vector $\mathbf{a}$ and any subset of its indexes S, we will write $\mathbf{a}_S$ to denote the sub-vector of $\mathbf{a}$ with indexes in S.

2. Model and coefficients of multivariate extremal dependence

Consider $\mathbf{X}_I = (X_1, \ldots, X_d)$ has df $F_{\mathbf{X}_I}$ and univariate marginals F_i such that

(i) $F_i(t) = \exp\left(-\sigma_i t^{-1/\eta}\right)$, $i = 1, \ldots, d$
(ii) $\ell_{\mathbf{X}_I}(t_1, \ldots, t_d) = -\ln F_{\mathbf{X}_I}(t_1, \ldots, t_d)$ is homogeneous of order $-1/\eta$,

for some constants $\sigma_i > 0$ and $\eta \in (0, 1]$. Thus, the copula $C_{\mathbf{X}_I}$ of $F_{\mathbf{X}_I}$ is max-stable, i.e.

$$C_{\mathbf{X}_I}(u_1^s, \ldots, u_d^s) = C^s_{\mathbf{X}_I}(u_1, \ldots, u_d), \ s > 0. \tag{1}$$

In the following we use notation $M(I) = \bigvee_{i\in I} F_i(X_i)$.

Lemma 2.1. *If $\mathbf{X}_I = (X_1, \ldots, X_d)$ satisfies conditions (i) and (ii) then, for all $(u_1, \ldots, u_p) \in (0, 1)^p$,*

$$P(M(I_1) \leq u_1, \ldots, M(I_p) \leq u_p) = \exp\left\{-\ell_{\mathbf{X}_I}\left(\sum_{j=1}^{p}\left(-\frac{\sigma_1}{\ln u_j}\right)^\eta \delta_1(I_j), \ldots, \sum_{j=1}^{p}\left(-\frac{\sigma_d}{\ln u_j}\right)^\eta \delta_d(I_j)\right)\right\},$$

where $\delta_i(I_j) = 1$ if $i \in I_j$ and $\delta_i(I_j) = 0$ otherwise. Analogously, we obtain, for $1 \leq j < j' \leq p$,

$$P(M(I_j) \leq u_j, M(I_{j'}) \leq u_{j'}) = \exp\left\{-\ell_{\mathbf{X}_{I_j\cup I_{j'}}}\left(\sum_{i\in\{j,j'\}}\left(-\frac{\sigma_{\alpha(I_j\cup I_{j'})}}{\ln u_i}\right)^\eta \delta_{\alpha(I_j\cup I_{j'})}(I_i), \ldots, \sum_{i\in\{j,j'\}}\left(-\frac{\sigma_{\omega(I_j\cup I_{j'})}}{\ln u_i}\right)^\eta \delta_{\omega(I_j\cup I_{j'})}(I_i)\right)\right\},$$

where $\alpha(I_j \cup I_{j'})$ and $\omega(I_j \cup I_{j'})$ denote the first and last points of $I_j \cup I_{j'}$, respectively.

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