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The residual extropy of order statistics

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ABSTRACT

Residual extropy was proposed to measure residual uncertainty of a random variable. Monotone properties and characterization results of this measure were studied. Similar properties of the proposed measure of order statistics were also discussed.

1. Introduction

Let X be a non-negative and absolutely continuous random variable with probability density function (pdf) f. To measure the uncertainty contained in X, the entropy was defined by Shannon (1948) as follows,

$$H(X) = -\int_0^\infty f(x)\log f(x)\,\mathrm{d}x.$$

Recently, an alternative measure of uncertainty called extropy was proposed by Lad et al. (2015). For random variable X, its extropy is defined as

$$J(X) = -\frac{1}{2}\int_0^\infty f^2(x)\,\mathrm{d}x. \tag{1.1}$$

One statistical application of extropy is to score the forecasting distributions. For example, under the total log scoring rule, the expected score of a forecasting distribution equals the negative sum of the entropy and extropy of this distribution (Gneiting and Raftery, 2007). In commercial or scientific areas such as astronomical measurements of heat distributions in galaxies, the extropy has been universally investigated (Furuichi and Mitroi, 2012; Vontobel, 2013). Most recently, Qiu (2017) further studied this new measure, exploring some characterization results, monotone properties and lower bounds of extropy of order statistics and record values.

As pointed out by Asadi and Ebrahimi (2000), if X is regarded as the lifetime of a new unit, then $H(X)$ is no longer useful for measuring the uncertainty about remaining lifetime of the unit. In such situations, one should consider the residual entropy

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of X, which was proposed by Ebrahimi (1996) as the entropy of $X_t = [X - t|X \geq t]$, i.e.,

$$H(X_t) = -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} \, dx, \tag{1.2}$$

where $\bar{F}$ is the survival function of X. Analogous to (1.2), the residual extropy of X is defined as the extropy of X_t in this paper. It is shown that the residual extropy of X is determined uniquely by its failure rate function in Section 2. Based on this point, several distributions are characterized in terms of its residual extropy. In Section 3, some monotone properties of residual extropy of the first order statistic are built. We also show that the underlying distributions can be characterized by the residual extropy of order statistics.

2. Residual extropy and characterization results

2.1. Residual extropy

By analogy to Ebrahimi (1996), we propose the following definition of the residual extropy. For random variable X, its residual extropy is defined as

$$J_t(X) \triangleq J(X_t) = -\frac{1}{2}\int_0^\infty f_t^2(x)\, dx = -\frac{1}{2\bar{F}^2(t)}\int_t^\infty f^2(x)\, dx, \quad t \geq 0, \tag{2.1}$$

where $f_t(x) = f(x+t)/\bar{F}(t),\ x \geq 0,\ t \geq 0$, is the pdf of X_t. It is obvious that the residual extropy of a continuous distribution is always negative, while the residual entropy of a continuous distribution may take any value on the extended real line, including $-\infty$ and ∞. It should be noted that if we put $t = 0$ in (2.1), then we get $J_0(X) = J(X)$, which coincides with (1.1).

The next theorem shows that the residual extropy of a random variable is determined uniquely by its failure rate function.

Theorem 2.1. *The residual extropy $J_t(X)$ of X is determined uniquely by $r_X(t)$, where $r_X(t) = f(t)/\bar{F}(t),\ t \geq 0$ is the failure rate function of X.*

Proof. It is obvious from (2.1) that

$$\begin{aligned} \frac{dJ_t(X)}{dt} &= -\frac{1}{2\bar{F}^4(t)}\left[-f^2(t)\bar{F}^2(t) + 2\bar{F}(t)f(t)\int_t^\infty f^2(x)\, dx\right] \\ &= \frac{1}{2}\left[r_X^2(t) + 4r_X(t)J_t(X)\right]. \end{aligned}$$

Thus, we have

$$\frac{dJ_t(X)}{dt} - 2r_X(t)J_t(X) = \frac{1}{2}r_X^2(t). \tag{2.2}$$

Solving the above differential equation leads to

$$J_t(X) = e^{2\int r_X(t)\,dt}\left[\frac{1}{2}\int r_X^2(t)e^{-2\int r_X(t)\,dt}\, dt + C\right], \tag{2.3}$$

where C is a constant and is determined by $J_t(X)|_{t=0} = J(X)$. This completes the proof. □

Remark 2.2. It follows from (2.2) that $J_t(X)$ is increasing (decreasing) in t if and only if $J_t(X) \geq (\leq) - r_X(t)/4$.

Example 2.3. Let X be a Pareto random variable with pdf $f(x) = \gamma\alpha^\gamma/(\alpha+x)^{\gamma+1},\ x \geq 0,\ \alpha, \gamma > 0$. Obviously, $r_X(x) = \gamma/(\alpha + x),\ x \geq 0$. It follows from (2.3) that

$$J_t(X) = e^{2\int \frac{\gamma}{\alpha+t}\,dt}\left[\frac{1}{2}\int \frac{\gamma^2}{(\alpha+t)^2}e^{-2\int \frac{\gamma}{\alpha+t}\,dt}\, dt + C\right] = -\frac{\gamma^2}{2(2\gamma+1)}\frac{1}{\alpha+t} + C(\alpha+t)^{2\gamma}, \quad t \geq 0.$$

Letting $t = 0$, we have

$$J_t(X)|_{t=0} = -\frac{\gamma^2}{2\alpha(2\gamma+1)} + C\alpha^{2\gamma} = J(X) = -\frac{\gamma^2}{2\alpha(2\gamma+1)}.$$

Thus, $C = 0$ and $J_t(X) = -\gamma^2/[2(2\gamma+1)(\alpha+t)],\ t \geq 0$. Obviously, $J_t(X)$ is increasing in t.

Table 1 lists the residual extropy (entropy) for some commonly-used distributions.

Next we will investigate alternative conditions under which $J_t(X)$ is decreasing in t. To this end, we first recall the definitions of two stochastic orders.