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ARCH model and fractional Brownian motion

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ABSTRACT

We study an extension of the ARCH model that includes the squared fractional Brownian motion. We study the statistical properties of the model as the conditions for the existence of a stationary solution and the moments of the process. We study their asymptotic behavior of the autocorrelation function of the squared of the process and we prove that the long memory property of the model holds. We illustrate our results by numerical simulations.

1. Introduction

The ARCH model has been introduced by Engle in Engle (1982) and then, it has been extended in many directions. One of the first extensions of the ARCH model, called the GARCH model, has been introduced by Bollerslev in Bollerslev (1986) and it also has been the object of various generalizations. We refer to Bollerslev (2008) for a glossary on the vast literature related with ARCH models and their generalizations.

The purpose of this work is to study an extension of the ARCH process which is able to capture the fluctuation of the intra-day price and the liquidity existent in the market. Recall that the ARCH(1) model is defined, for every $t \in \mathbb{Z}$, by

$$X_t = \sigma_t \varepsilon_t$$

with

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 \tag{1}$$

where $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a sequence of i.i.d. random variable such that $E\varepsilon_0 = 0$ and $E\varepsilon_0^2 = 1$ and $\alpha_0 > 0, \alpha_1 \geq 0$.

In order to capture the fluctuation of the intra-day price in financial markets, we include in the model the price range for a financial asset in a given trading day, that is the difference between the maximum price (denoted h_t or the highest price at lag t or trading day t) and the minimum (lowest) price or m_t during the same trading day t. As a proxy for the liquidity we employ the number of shares traded during the trading day t or the trading volume, denoted L_t. This variable is added in the formula for the volatility, see relation (3) in the next section of our paper. Therefore, our model weighs the impact of past shocks with their corresponding liquidity. In other words, if the past volatility was accompanied by high liquidity, then its impact on the future volatility will be larger. The model therefore does not allow the "false shocks" to have a high explanatory power, considering that the lack of liquidity creates a distorted picture of reality, and shocks that occur under such conditions must be corrected for their low liquidity. To better illustrate this, let us consider a certain asset A and let us imagine the hypothetical situation where during a certain trading day t only two transactions with asset A took place at large

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time intervals and with low trading volume, but with a large value for $|h_t - m_t|$ (or the difference between the maximum price and the minimum (lowest) price), that is the two transactions took place at two significantly different prices. Consider also that in another trading session $t+i$, the asset A has also traded in the range of prices $[h_{t+i}, m_{t+i}]$ (where $h_{t+i} = h_t$ and $m_{t+i} = m_t$), but this time the transactions were numerous and the trading volume was high. Obviously, the trading day $t+i$ was more "turbulent" than the day t. But if in both cases only $|h_t - m_t|$ is considered as a measure of volatility without including the corresponding liquidity, then the shocks produced at time t and $t+i$ will have an identical impact on the future volatility, which would be clearly erroneous, as the shock from $t+i$ is much stronger than the shock from t. See also Tudor and Tudor (2014) for an empirical study of a EGARCH model with weighted liquidity.

We will model the liquidity at time t, denoted L_t in the sequel, by the square of the increment of the fractional Brownian motion. That is, we will set $L_t = \ell_1 \left(B^H_{t+1} - B^H_t\right)^2$ for every $t, s \in \mathbb{Z}$ where ℓ_1 is a strictly positive constant and $(B^H_t)_{t\in\mathbb{Z}}$ denotes a fractional Brownian motion with Hurst exponent $H \in (0, 1)$. Recall that the fractional Brownian motion (fBm in the sequel) is defined as a centered Gaussian process with covariance

$$EB^H_t B^H_s = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \qquad t, s \in \mathbb{Z}.$$

This process is the only Gaussian self-similar process with stationary increments and it has long memory for $H > \frac{1}{2}$. In the last decades, the stochastic analysis for the fBm has known a huge development due to the various applications of the fBm.

Our choice to model the liquidity by the squared increment of the fBm is determined by several reasons. First, the empirical data reveal the existence of the long-memory property in liquidity (see e.g. Tsuji (2002)), therefore fBm appears as a good candidate to model it. Second, in order to ensure that the squared volatility (given by the expression (3) remains positive, we choose the squared increment of the fBm. The fact that the increments of the fBm are not independent changes in a radical way the structure of the model, see Section 2. The Hurst parameter of the fractional Brownian motion will affect the moments and the covariance of the ARCH process, as well as the asymptotic behavior of the estimators for the parameters of the model. Although the parameter estimation will be the object of a future work, let us mention that the standard least squares estimator (LSE) as constructed in e.g. Francq and Zakoïan (2010) does not work in our model. It can be actually shown that the standard LSE for the parameter α_1 is not consistent if $H \neq \frac{1}{2}$.

We organized our paper as follows. In Section 2, the ARCH model with weighted liquidity is presented. Section 3 is devoted to analyze the behavior of the correlation function of the process. A simulation study is reported in the last section.

2. The ARCH model with weighted liquidity

We introduce our variant of the ARCH model with fBm innovations and we analyze its dependence structure. The model is defined as follows: for every $t \in \mathbb{Z}$

$$X_t = \sigma_t \varepsilon_t \tag{2}$$

with

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \ell_1 L_{t-1}. \tag{3}$$

Here we assume $\alpha_0 \geq 0, \alpha_1 > 0$ and $\ell_1 > 0$. We assume that $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a sequence of i.i.d. random variable such that $E\varepsilon_0 = 0$ and $E\varepsilon_0^2 = 1$. The sequence $(\varepsilon_t)_{t\in\mathbb{Z}}$ is referred to as the driving noise sequence. The parameters $\alpha_0, \alpha_1, \ell_1$ are assumed to be strictly positive. The sequence $(L_t)_{t\in\mathbb{Z}}$ is a sequence of identically distributed positive random variables; in particular $EL_t = 1$ for every $t \in \mathbb{Z}$. In addition we will assume that it is independent of the sequence $(\varepsilon_t)_{t\in\mathbb{Z}}$.

As mentioned in the introduction, we will assume that the new element L_t in the model is the square of the increment of the fractional Brownian motion, that is, for every $t \in \mathbb{Z}$,

$$L_t := L^H_t = \left(B^H_{t+1} - B^H_t\right)^2 \tag{4}$$

where $(B^H_t)_{t\in\mathbb{Z}}$ denotes the fractional Brownian motion with Hurst parameter $H \in (0, 1)$. Note that the sequence $(L_t)_{t\in\mathbb{Z}}$ is not independent since the fBm has dependent increments. The case $\ell_1 = 0$ corresponds to the classical ARCH(1) model introduced in Engle (1982).

2.1. The existence of the stationary solution

The first step is to prove the existence of a stationary solution to the problem (2)–(3). From (2) and (3) we can immediately write, for every $t \in \mathbb{Z}$,

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 \sigma_{t-1}^2 + \ell_1 L_{t-1}. \tag{5}$$

We introduce the following notation:

$$y_t := \sigma_{t+1}^2, \quad A_t := \alpha_1 \varepsilon_t^2, \quad B_t = \alpha_0 + \ell_1 L_t, \qquad t \in \mathbb{Z} \tag{6}$$

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