



A strong law of large numbers for sub-linear expectation under a general moment condition

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ABSTRACT

In this paper, we derive a strong law of large numbers for sub-linear expectation under a general moment condition. The result can be reduced to the classical strong law of large numbers when the sub-linear expectation coincides with the classical linear expectation. Moreover, we illustrate that this moment condition for strong law of large numbers in sub-linear situation is the weakest.

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1. Introduction

The classical strong laws of large numbers (SLLNs for short) are widely known as fundamental results in probability theory whose importance requires no discussion. Recently, motivated by some problems in statistics, measures of risk, mathematical economics and super-hedging in finance, more and more researches adopted non-additive probability and non-linear expectation to describe some uncertain phenomena in these fields which cannot be modeled exactly by classical probability theory; for example [Chen and Epstein \(2002\)](#), [Choquet \(1953\)](#), [Huber and Strassen \(1973\)](#), [Peng \(1997, 2010\)](#) and [Wasserman and Kadane \(1990\)](#). Nowadays the SLLNs for non-additive probability and non-linear expectation are widely studied by a number of researchers; for instance [Chen et al. \(2013\)](#), [Cozman \(2010\)](#), [Korchevsky \(2015\)](#), [Li and Chen \(2011\)](#), [Marinacci \(1999\)](#), [Maccheroni and Marinacci \(2005\)](#) and [Zhang \(2016\)](#).

The general sub-linear expectation and related notions of independent and identically distributed random variables are introduced by [Peng \(2010\)](#). The previous results of SLLNs for sub-linear expectation showed that given a sequence $\{X_n\}_{n=1}^{\infty}$ of i.i.d. random variables for sub-linear expectation, any cluster point of empirical averages lies between the lower expectation (super-linear expectation) $\mathcal{E}[X_1]$ and the upper expectation (sub-linear expectation) $\mathbb{E}[X_1]$ quasi-surely. That is

$$v\left(\mathcal{E}[X_1] \leq \liminf_{n \rightarrow \infty} \frac{S_n}{n} \leq \limsup_{n \rightarrow \infty} \frac{S_n}{n} \leq \mathbb{E}[X_1]\right) = 1.$$

[Chen et al. \(2013\)](#) obtained a SLLN of the above form for independent random variables under the condition of finite $(1+\alpha)$ th moment for upper expectation. [Zhang \(2016\)](#) derived a SLLN of this form for negatively dependent identically distributed random variables under the condition of finite 1st moment for Choquet expectation. As we all know the Choquet expectation with respect to a sub-additive capacity is much larger than the sub-linear expectation. So it would be interesting to know whether we can obtain a SLLN under some weaker moment condition for sub-linear expectation.

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In this paper, we need the following notations. Let Φ_c (or, respectively, Φ_d) denote the set of functions $\psi(x)$ defined on $[0, \infty)$ such that $\psi(x)$ satisfies:

- (1) $\psi(x)$ is nonnegative and nondecreasing on $[0, \infty)$ and positive on $[x_0, \infty)$ for some $x_0 \geq 0$. The series $\sum_{n=[x_0]+1}^{\infty} \frac{1}{n\psi(n)}$ converges (respectively, diverges).
- (2) For any fixed $a > 0$, there exists $C > 0$ such that $\psi(x+a) \leq C\psi(x)$ for any $x \geq x_0$.

The value of x_0 is not assumed to be the same for different ψ . Functions x^α and $(\ln(1+x))^{1+\alpha}$ for any $\alpha > 0$ are examples of the class of functions Φ_c . The functions $\ln(1+x)$ and $\ln \ln(e \vee x)$ belong to the class Φ_d .

The purpose of this paper is to show that the SLLN is still true under a general moment condition which is weaker than finite $(1+\alpha)$ th moment. Inspired by [Petrov \(1975\)](#) and [Korchevsky \(2015\)](#), we conjecture and then prove that the moment condition $\sup_n \mathbb{E}[|X_n| \psi(|X_n|)] < \infty$ for some $\psi \in \Phi_c$ can maintain the validity of the SLLN and this moment condition is the weakest. That is when only $\sup_n \mathbb{E}[|X_n| \psi(|X_n|)] < \infty$ for some $\psi \in \Phi_d$ holds, we can always find a counterexample such that the SLLN is not true.

The rest of this paper is organized as follows. In Section 2, we recall some basic concepts and lemmas related to sub-linear expectation. In Section 3, we state and prove our main result, that is the SLLN under a general moment condition for sub-linear expectation. In Section 4, we illustrate that this condition is the weakest.

2. Basic concepts and lemmas

We use the notations similar to that of [Peng \(2010\)](#). Let $(\Omega, \mathcal{F})$ be a given measurable space. Let $\mathcal{H}$ be a linear space of real functions defined on Ω such that if $X_1, X_2, \dots, X_n \in \mathcal{H}$ then $\varphi(X_1, X_2, \dots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$ where $C_{l,Lip}(\mathbb{R}^n)$ denotes the linear space of (local Lipschitz) function φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq C(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

for some $C > 0, m \in \mathbb{N}$ depending on φ . $\mathcal{H}$ contains all I_A where $A \in \mathcal{F}$. We consider $\mathcal{H}$ as the space of random variables.

Definition 2.1. A sub-linear expectation $\mathbb{E}$ on $\mathcal{H}$ is a functional $\mathbb{E}: \mathcal{H} \rightarrow \overline{\mathbb{R}} := [-\infty, \infty]$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

- (a) Monotonicity: If $X \geq Y$ then $\mathbb{E}[X] \geq \mathbb{E}[Y]$.
- (b) Constant preserving: $\mathbb{E}[c] = c, \forall c \in \mathbb{R}$.
- (c) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \forall \lambda \geq 0$.
- (d) Sub-additivity: $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$ whenever $\mathbb{E}[X] + \mathbb{E}[Y]$ is not of the form $\infty - \infty$ or $-\infty + \infty$.

Remark 2.1. By combining (b) and (d) we can easily obtain a basic property of sub-linear expectation:

- (e) Translation invariance: $\mathbb{E}[X + c] = \mathbb{E}[X] + c, \forall c \in \mathbb{R}$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sub-linear expectation space. Given a sub-linear expectation $\mathbb{E}$, we define the conjugate expectation $\mathcal{E}$ of $\mathbb{E}$ by

$$\mathcal{E}[X] := -\mathbb{E}[-X], \quad \forall X \in \mathcal{H}.$$

Obviously, for all $X \in \mathcal{H}, \mathcal{E}[X] \leq \mathbb{E}[X]$.

Definition 2.2. A set function $V: \mathcal{F} \rightarrow [0, 1]$ is called a capacity if it satisfies

- (a) $V(\emptyset) = 0, V(\Omega) = 1$.
- (b) $V(A) \leq V(B), A \subset B, A, B \in \mathcal{F}$.

A capacity V is said to be sub-additive if it satisfies $V(A \cup B) \leq V(A) + V(B), A, B \in \mathcal{F}$.

The corresponding Choquet expectation is defined by

$$C_V[X] := \int_0^\infty V(X \geq t) dt + \int_{-\infty}^0 [V(X \geq t) - 1] dt.$$

In this paper we only consider the capacity induced by sub-linear expectation. Given a sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, we define a capacity: $V(A) := \mathbb{E}[I_A], \forall A \in \mathcal{F}$. We also define the conjugate capacity: $v(A) := 1 - V(A^c), \forall A \in \mathcal{F}$. Clearly, we can easily obtain that V is a sub-additive capacity and $v(A) = \mathcal{E}[I_A]$.

Definition 2.3. A sub-linear expectation $\mathbb{E}: \mathcal{H} \rightarrow \mathbb{R}$ is said to be continuous if it satisfies:

- (1) Lower-continuity: if $X_n \uparrow X$, then $\mathbb{E}[X_n] \uparrow \mathbb{E}[X]$, where $0 \leq X_n, X \in \mathcal{H}$.
- (2) Upper-continuity: if $X_n \downarrow X$, then $\mathbb{E}[X_n] \downarrow \mathbb{E}[X]$, where $0 \leq X_n, X \in \mathcal{H}$.

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