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A note on martingale deviation bounds

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ABSTRACT

Let $\{Y_i\}_{i=1}^n$ be a martingale difference sequence and $S_n = \sum_{i=1}^n Y_i$. Probability deviation bounds for martingale difference sequences generally focus on upper bounds for probabilities of large deviations $P(S_n > \lambda)$, particularly of maxima of S_n. In this article bounds for probabilities of moderate deviations $P(S_n < \lambda)$ are studied. The motivation is estimating the probability that the cumulative drift of a Markov chain is moderate, and thus estimates derived from sampling the chain are reliable.

1. Introduction

Let $\{Y_i\}_{i=1}^n$ be a martingale difference sequence with $E|Y_i|^p = M$ for some $p > 1$. Let $S_n = \sum_{i=1}^n Y_i$. A good deal of work (see for example Lesigne and Volny, 2001; Li, 2003) has gone into developing upper bounds for probabilities of large deviations for maxima of sums of the martingale difference sequence, i.e., for a given $\lambda > 0$ interest is in bounds for

$$P\left(\max_{1\le i\le n} |S_i| > \lambda\right).$$

This article concerns upper bounds for probabilities of moderate, rather than large, deviations, i.e., bounds for

$$P\left(\max_{1\le i\le n} |S_i| < \lambda\right).$$

Lower bounds can be obtained from existing results which provide upper bounds for

$$P\left(\max_{1\le i\le n} |S_i| > \lambda\right).$$

The motivation for studying moderate deviations comes from Markov chain applications. The cumulative drift of a Markov chain can be expressed as the sum of a martingale difference sequence and two other terms. When the cumulative drift of a Markov chain is moderate the Markov chain is near a stable region in the state-space and sampling can occur with an eye on, for example, producing good estimates of $E[f(X_t)]$ using $1/n\sum_{i=0}^{n-1} f(X_i)$ (see Kontoyiannis et al., 2006). Thus, it is worthwhile to monitor the drift of the chain in order to improve the effectiveness of MCMC techniques, as one example application.

Specifically, let $\{X_t\}$ be a time-homogeneous discrete time Markov chain. For a function $V \ge 1$ the drift of the process at each transition with respect to V is given by $E[V(X_{i+1})|X_i] - V(X_i)$ and the cumulative drift of the process is

$$S_n(V) = \sum_{i=0}^{n-1} E[V(X_{i+1})|X_i] - V(X_i) = \sum_{i=0}^{n-1} E[V(X_{i+1})|X_i] - V(X_{i+1}) - V(X_0) + V(X_n),$$

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where $\sum_{i=0}^{n-1} E[V(X_{i+1})|X_i] - V(X_{i+1})$ is the sum of the martingale difference sequence $\{E[V(X_{i+1})|X_i] - V(X_{i+1})\}_{i=0}^{n-1}$. The object then is to bound the probability the cumulative drift is moderate

$$P\left(\max_{1\le i\le n} |S_i(V)| < \lambda\right),$$

as a complement to the conditional probability the estimate is near the target given the moderate drift; i.e., for $\epsilon > 0$

$$P\left(\left|\frac{1}{n}\sum_{i=0}^{n-1} f(X_i) - E[f(X_t)]\right| > n\epsilon \,\Big|\, \max_{1\le i\le n} |S_i(V)| < \lambda\Big|\right).$$

This being said, the emphasis on Markov chain applications will be suppressed in this article in favor of speaking generally about martingale difference sequences.

2. Results

The main result depends upon the following lemmas which appear in Li (2003) and borrow from Hall and Heyde (1980) and Lesigne and Volny (2001). The lemmas use arguments based on Burkholder's inequality for L^p martingales (see Hall and Heyde, 1980) and convexity to arrive at the results. Both lemmas establish bounds on $E|S_n|^p$, treating the cases $1 < p < 2$ and $p > 2$ separately.

Lemma 1. *Let $\{X_i\}_{i=1}^n$ be a finite sequence of a martingale difference where $X_i \in L^p$ for some $p \in (1, 2]$, and let $S_n = \sum_{i=1}^n X_i$. Then*

$$E|S_n|^p \le b_p^p \sum_{i=1}^n E|X_i|^p,$$

$$E|S_n|^p \ge a_p^{-p} n^{p/2-1} \sum_{i=1}^n E|X_i|^p,$$

where $b_p = 18pq^{1/2}$, $a_p = 18p^{1/2}q$ and q is such that $1/p + 1/q = 1$.

Slightly different results are to be expected when $p > 2$. When $p = 2$ establishing bounds on the probabilities of deviations are not difficult.

Lemma 2. *Let $\{X_i\}_{i=1}^n$ be a finite sequence of a martingale difference where $X_i \in L^p$ for some $p > 2$, and let $S_n = \sum_{i=1}^n X_i$. Then*

$$E|S_n|^p \le n^{p/2-1} b_p^p \sum_{i=1}^n E|X_i|^p,$$

$$E|S_n|^p \ge a_p^{-p} \sum_{i=1}^n E|X_i|^p,$$

where $b_p = 18pq^{1/2}$, $a_p = 18p^{1/2}q$ and q is such that $1/p + 1/q = 1$.

The following is the main result. The previous lemmas are applied to establish bounds on the probabilities of moderate deviations in the separate cases $1 < p < 2$ and $p > 2$.

Theorem 1. *Let $\{Y_i\}_{i=1}^n$ denote a finite martingale difference sequence. Let $S_n = \sum_{i=1}^n Y_i$. Suppose $E|Y_i|^p = M < \infty$, $(E|Y_i|^{p+\delta})^{1/(p+\delta)} < \infty$ for some $\delta > 0$.*

(i) *If $1 < p < 2$, then*

$$P\left(\max_{i\in\{1,\ldots,n\}} |S_i| \le \lambda\right) \le 1 - \left(\frac{c_n E|S_n|^p - \lambda^p}{[\lambda + M]^p}\right)^{(p+\delta)/\delta},$$

for $\lambda > 0$ such that

$$2^{-1/p}(c_n E|S_n|^p)^{1/p} - M/2 \le \lambda < (c_n E|S_n|^p)^{1/p},$$

where $c_n = [\prod_{i=1}^{n-1}(1 + a_p/\sqrt{n-i})]^{-p}$ for $n \ge 2$, $c_1 = 1$, and $a_p = 18p^{3/2}/(p-1)$.

(ii) *If $p > 2$, then*

$$P\left(\max_{i\in\{1,\ldots,n\}} |S_i| \le \lambda\right) \le 1 - \left(\frac{c_n E|S_n|^p - \lambda^p}{[\lambda + M]^p}\right)^{(p+\delta)/\delta},$$