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Q1 # Limit theorems for order statistics from exponentials[☆]

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ABSTRACT

In this paper we study the ratio of various order statistics based on samples from an exponential distribution and establish a central limit theorem and the almost sure central limit theorem for these statistics.

1. Introduction

Let X_{ni} be exponential random variables with mean λ_n, where $i = 1, 2, \ldots, m_n$ and $n = 1, 2, 3, \ldots$. Here m_n denotes the sample size, and the model can be interpreted as sampling lifetimes of a machine and we can change the equipment on a daily basis. Let the order statistics be $X_{n(1)} \le X_{n(2)} \le \cdots \le X_{n(m_n)}$, and an interesting statistics is the ratios of these order statistics, i.e.,

$$R_{nij} = \frac{X_{n(j)}}{X_{n(i)}}, \quad 1 \le i < j \le m_n.$$

There are several interesting ratios R_{nij} that need to be examined, for example, the parameters i, j can change as previously noted. The most important statistics are R_{n12} and R_{n23}. They can measure the stability of our equipment and they show whether or not our system is stable, since the exponential random variables measure the lifetimes of equipment.

The statistics R_{n12} is the measure of the failure of the first piece of our equipment. Comparing the smallest order statistic to the others tells us how stable our system is. Adler (2015) studied the strong laws of the ratio R_{n12} under the cases that Q3 the sample size m_n is fixed or $m_n \to \infty$ as follows: for all $\alpha > -2$, if $m_n = m$ is fixed, then we have

$$\lim_{N\to\infty} \frac{1}{(\log N)^{\alpha+2}} \sum_{n=1}^{N} \frac{(\log n)^{\alpha} X_{n(2)}}{n X_{n(1)}} = \frac{m}{(m-1)(\alpha+2)}, \ a.s.$$

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if $m_n \to \infty$, then we have

$$\lim_{N\to\infty} \frac{1}{(\log N)^{\alpha+2}} \sum_{n=1}^{N} \frac{(\log n)^{\alpha} X_{n(2)}}{n X_{n(1)}} = \frac{1}{\alpha+2}, \ a.s.$$

The properties of the statistic R_{n23} are different from R_{n12}, for instance, the mean of R_{n12} is infinite and the mean of R_{n23} is finite.

In the present paper, we are interested in the statistic R_{n2j}. For convenience sake, we first study the statistic R_{n23}, and for the statistic R_{n2j}, the proof is similar. In the following section, we shall establish the central limit theorem and the almost sure central limit theorem for the statistic R_{n23}.

2. Main results

First we give the density function of R_{n23} with fixed sample size $m_n = m$. For every $n \geq 1$, let $\{X_{ni}, i \geq 1\}$ be a sequence of independent exponential random variables with mean λ_n, and let $\{X_n, n \geq 1\} := \{(X_{ni}, i \geq 1), n \geq 1\}$ be independent random sequences. The order statistics are $X_{n(1)} \leq X_{n(2)} \leq \cdots \leq X_{n(m)}$. The joint density of the second and third order statistics $X_{n(2)}, X_{n(3)}$ is

$$f_n(x_2, x_3) = \begin{cases} \dfrac{m!}{(m-3)!\lambda_n^2} e^{-x_2/\lambda_n} e^{-x_3(m-2)/\lambda_n} (1 - e^{-x_2/\lambda_n}), & 0 < x_2 < x_3 \\ 0, & \text{otherwise.} \end{cases}$$

We transform to the variables $\omega = x_2$ and $r = x_3/x_2$. The Jacobian is ω and the joint density of ω and r is

$$f_n(\omega, r) = \begin{cases} \dfrac{m!}{(m-3)!\lambda_n^2} \omega e^{-\omega[1+r(m-2)]/\lambda_n} (1 - e^{-\omega/\lambda_n}), & \omega > 0, r > 1. \\ 0, & \text{otherwise.} \end{cases}$$

Thus the density function of $R_{n23} := X_{n(3)}/X_{n(2)}$ is

$$f(x) = \frac{m!}{(m-3)!} \left[\left(\frac{1}{1 + x(m-2)} \right)^2 - \left(\frac{1}{2 + x(m-2)} \right)^2 \right], \quad x \geq 1.$$

Adler (2015) obtained the expectation of R_{n23}

$$\mathbb{E}(R_{n23}) = 1 + \frac{m(m-1)}{m-2} \log \left(\frac{m}{m-1} \right).$$

Our first result is about the asymptotic distribution of the sums of $\sum_{n=1}^{N} R_{n23}$. The following lemma gives some properties for the slowly varying function at ∞.

Lemma 2.1 (Csörgő et al., 2003, *Lemma 1*). *Let ξ be a random variable with $\mathbb{E}\xi = 0$, and let*

$$L(x) := \mathbb{E}\xi^2 1_{\{|\xi| \leq x\}}, \tag{2.1}$$

then the following statements are equivalent:

(a) $x^2 \mathbb{P}(|\xi| > x) = o(L(x))$;
(b) $x\mathbb{E}|\xi| 1_{\{|\xi| > x\}} = o(L(x))$;
(c) $\mathbb{E}|\xi|^{\alpha} 1_{\{|\xi| \leq x\}} = o(x^{\alpha-2} L(x))$ *for* $\alpha > 2$;
(d) $L(x)$ *is a slowly varying function at* ∞, *i.e.*, $L(cx) \sim L(x)$ *as* $x \to \infty$ *for each* $c > 0$.

When $L(x)$ is a slowly varying function at ∞, a well-known result is $\mathbb{E}|\xi|^p < \infty$ for all $0 \leq p < 2$. Moreover, if $L(x) := \mathbb{E}\xi^2 1_{\{|\xi| \leq x\}}$ varies slowly at ∞, then so does the function $L_m(x) := \mathbb{E}(\xi - m)^2 1_{\{|\xi - m| \leq x\}}$ for every $m \in \mathbb{R}$. Let

$$\eta_n = 1 \vee \sup\{r > 0; nL(r) \geq r^2\}, \quad n \in \mathbb{N},$$

then it is easy to check that

$$\eta_n \to \infty \quad \text{and} \quad \eta_n^2 \sim nL(\eta_n).$$

Lemma 2.2 (Kallenberg, 1997, *Theorem 4.17*). *Let $\xi_1, \xi_2, \ldots$ be i.i.d. nondegenerate random variables, and let ζ be $N(0, 1)$. Then $a_n \sum_{k \leq n} (\xi_k - m_n)$ for some constants a_n and m_n if and only if the function $L(x) := \mathbb{E}\xi^2 1_{\{|\xi| \leq x\}}$ varies slowly at ∞, in which case we may take $m_n = 0$. (From the proof of* Kallenberg (1997, Theorem 4.17), *a_n can be taken by η_n^{-1}).*