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Small ball probabilities for a class of time-changed self-similar processes

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ABSTRACT

This paper establishes small ball probabilities for a class of time-changed processes $X \circ E$, where X is a self-similar process and E is an independent continuous process, each with a certain small ball probability. In particular, examples of the outer process X and the time change E include an iterated fractional Brownian motion and the inverse of a general subordinator with infinite Lévy measure, respectively. The small ball probabilities of such time-changed processes show power law decay, and the rate of decay does not depend on the small deviation order of the outer process X, but on the self-similarity index of X.

1. Introduction

Let W be a one-dimensional standard Brownian motion and let E_β be the inverse of a stable subordinator D_β of index $\beta \in (0, 1)$, independent of W. Nane established in Nane (2009) that the small ball probability of the time-changed Brownian motion $W \circ E_\beta$ is given by

$$\mathbb{P}\Big(\sup_{0\le t\le 1} |W(E_\beta(t))| \le \epsilon\Big) \sim \frac{32\Gamma(\beta)\sin(\beta\pi)}{\pi^4} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^3}\epsilon^2 \quad \text{as } \epsilon \downarrow 0, \tag{1}$$

where $\Gamma(\cdot)$ is Euler's Gamma function and the notation $f(x) \sim g(x)$ means that $\lim f(x)/g(x) = 1$. The result is interesting since the small ball probability of $W \circ E_\beta$ shows power law decay unlike the exponential decay observed for the Brownian motion W: $-\log \mathbb{P}(\sup_{0\le t\le 1} |W(t)| \le \epsilon) \sim (\pi^2/8)\,\epsilon^{-2}$. Moreover, the rate of decay in (1) does not depend on the stability index β of the underlying stable subordinator D_β; the dependence on β only appears as a small deviation constant.

The proof of (1) provided in Nane (2009) essentially relies on the following expression for the Laplace transform of the random variable $E_\beta(1)$ and its asymptotic behavior along the negative real axis (see e.g. Proposition 1(a) of Bingham, 1971 and Theorem 1.4 of Podlubny, 1999): $\mathbb{E}[e^{-aE_\beta(1)}] = \mathbf{E}_\beta(-a) \sim 1/[a\Gamma(1-\beta)]$ as $a \to \infty$. Here, $\mathbf{E}_\beta(z) = \sum_{n=0}^{\infty} z^n/\Gamma(n\beta+1)$ is the Mittag-Leffler function. That paper also extended the result to a time-changed process $X \circ E_\beta$, where the outer process

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X is a self-similar process possessing a certain small ball probability, which particularly includes the case of a fractional Brownian motion. However, the exact small deviation constant cannot be specified unlike the situations considered in Aurzada and Lifshits (2009); see Remark 5 for details of this point.

In this paper, we establish small ball probabilities for a class of time-changed processes $X \circ E$, where X is a self-similar process and E is a continuous process independent of X, each with a certain small ball probability (Theorems 1 and 2). This largely extends the results in Nane (2009) in terms of both the outer process X and the time change E. Examples of X and E that can be handled within our framework include an iterated fractional Brownian motion and the inverse of a general subordinator with infinite Lévy measure having no atoms, respectively. Our strategy is to employ a version of the Tauberian theorem (Lemma 1) along with a general fact concerning the inverse of a subordinator (Proposition 1), which is a different approach from what was taken in Nane (2009) to derive (1). In particular, even when E is the inverse of a stable subordinator, our proof does not rely on the asymptotic expression of the Mittag-Leffler function.

The results to be established in this paper show that the small ball probability of a certain time-changed process $X \circ E$ has power law decay whose rate depends on the self-similarity index of the outer process X, but not on the small deviation order of X. In a particular case of a time-changed Brownian motion $W \circ E$ with the time change E being the inverse of a general subordinator with infinite Lévy measure, the dependence on E is reflected on the associated small deviation constant. We will specify that constant when the underlying subordinator is a Gamma subordinator or a tempered stable subordinator; these specific time changes have been recently investigated to analyze anomalous diffusions observed in various natural phenomena (see e.g. Janczura and Wiłomańska, 2012). This will allow us to examine how the small ball probabilities for the important subclasses of time-changed processes vary according to the choice of the parameters defining the underlying subordinators. We note that small deviation probabilities of time-changed processes are also investigated in Frolov (2013a,b), but the results presented in those papers show exponential decay unlike the power law decay observed in our results.

The main motivation to analyze such time-changed processes comes from their non-standard structures. In particular, the time-changed Brownian motion $W \circ E_\beta$ is non-Gaussian and non-Markovian, and is widely used to model subdiffusions, where particles spread more slowly than the classical Brownian particles do. One interesting aspect of the time-changed Brownian motion is that its transition probabilities satisfy the following time-fractional generalization of the Fokker–Planck or forward Kolmogorov equation: $\partial_t^\beta p(t,x) = \frac{1}{2}\partial_x^2 p(t,x)$, where ∂_t^β denotes the Caputo fractional derivative operator in time of order β (see e.g. Podlubny, 1999). Nigmatullin (1986) provided a physical derivation of a time-fractional Kolmogorov equation which involves the generator of some continuous Markov process. Since then various kinds of time-changed processes as well as their associated fractional equations have been investigated; see e.g. Hahn et al. (2011a,b, 2012), Kobayashi (2011), Magdziarz (2009b), Magdziarz and Zorawik (in press), Meerschaert and Scheffler (2004, 2008), and Orsingher and Beghin (2004, 2009). Fractional generalizations of Cauchy problems on bounded domains with Dirichlet boundary conditions are discussed in Chen et al. (2011) and Meerschaert et al. (2009, 2011). Fractional Kolmogorov equations have found many applications in a wide range of scientific areas; see e.g. Benson et al. (2000), Gorenflo et al. (2001), Magdziarz (2009a), Metzler and Klafter (2000), Saxton and Jacobson (1997) and Zaslavsky (1994).

2. Small ball probabilities for time-changed Brownian motions

Let D be a subordinator with Laplace exponent ψ and infinite Lévy measure ν; i.e. D is a one-dimensional nondecreasing Lévy process with càdlàg paths starting at 0 with Laplace transform $\mathbb{E}[e^{-sD(t)}] = e^{-t\psi(s)}$, where $\psi(s) = bs + \int_0^\infty (1 - e^{-sx})\,\nu(dx)$. Here, $b \geq 0$ and $\int_0^\infty (x \wedge 1)\,\nu(dx) < \infty$. The assumption that the Lévy measure is infinite (i.e. $\nu(0,\infty) = \infty$) implies that ψ is an increasing function with $\lim_{s\to\infty}\psi(s) = \infty$ and D has strictly increasing paths with infinitely many jumps (see e.g. Theorem 21.3 of Sato, 1999). Let E be the *inverse* or *first hitting time process* of D; i.e. $E(t) := \inf\{u > 0; D(u) > t\}$ for $t \geq 0$. Since D has strictly increasing paths, the process E, called an *inverse subordinator*, has continuous, nondecreasing paths starting at 0 (see e.g. Lemma 2.7 of Kobayashi, 2011). It is known that E generally does not have independent or stationary increments (see Section 3 of Meerschaert and Scheffler, 2004), which implies that even if X is a Gaussian or Lévy process independent of E, the time-changed process $X \circ E$ no longer has the same structure as X. Hence, existing results on small ball probabilities of Gaussian or Lévy processes cannot be directly applied to find the small ball probability of $X \circ E$.

The following theorem extends (1) to the case when the time change is a general inverse subordinator.

Theorem 1. *Let E be the inverse of a subordinator D with infinite Lévy measure ν, independent of a one-dimensional standard Brownian motion W. Then for all $T > 0$ at which ν has no mass (i.e. $\nu(\{T\}) = 0$),*

$$\mathbb{P}\Big(\sup_{0\leq t\leq T} |W(E(t))| \leq \epsilon\Big) \sim \frac{32}{\pi^3}\nu(T,\infty)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^3}\epsilon^2 \quad \text{as } \epsilon \downarrow 0. \tag{2}$$

If $\nu(T,\infty) = 0$, this is interpreted as $\mathbb{P}(\sup_{0\leq t\leq T}|W(E(t))| \leq \epsilon) = o(\epsilon^2)$.

Remark 1. (1) This theorem shows that, if $\nu(T,\infty) > 0$, then the small ball probability of the time-changed Brownian motion $W \circ E$ has a power law decay. Moreover, the rate of decay of the small ball probability does not depend on the choice of the inverse subordinator E; the dependence on E is reflected only on the constant $\nu(T,\infty)$.