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Some integrals involving multivariate Hermite polynomials: Application to evaluating higher-order local powers

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ABSTRACT

We present the formula for a certain integral with respect to multivariate Hermite polynomials. Such integrals are used for deriving higher-order local power functions of asymptotically chi-squared tests. As an example, we provide asymptotic expansion for the local power function of Rao's score test.

1. Introduction

Let $\phi_\Sigma(\cdot)$ be the density of the p-variate normal distribution $\mathrm{N}_p(\mathbf{0}_p, \mathbf{\Sigma})$;

$$\phi_\Sigma(\mathbf{v}) = \frac{1}{(2\pi)^{p/2}|\mathbf{\Sigma}|^{1/2}} \exp\Big(-\frac{\mathbf{v}'\mathbf{\Sigma}^{-1}\mathbf{v}}{2}\Big), \quad \mathbf{v} = (v_1, \ldots, v_p)',$$

where $\mathbf{\Sigma} = [\sigma_{j,j'}]_{j,j'\in\{1,\ldots,p\}}$ is a $p \times p$ positive definite matrix. For any $n \in \mathbf{N}$, we set

$$\phi_\Sigma^{j_1,\ldots,j_n}(\mathbf{v}) = (-1)^n\Big(\prod_{i=1}^{n} \frac{\partial}{\partial v_{j_i}}\Big)\phi_\Sigma(\mathbf{v}), \quad j_1, \ldots, j_n \in \{1, \ldots, p\};$$

$\phi_\Sigma^{j_1,\ldots,j_n}(\mathbf{v})/\phi_\Sigma(\mathbf{v})$ is referred to as the multivariate Hermite polynomial. For example,

$$\begin{aligned}
\phi_\Sigma^{j_1}(\mathbf{v}) &= \widetilde{v}_{j_1}\phi_\Sigma(\mathbf{v}),\\
\phi_\Sigma^{j_1,j_2}(\mathbf{v}) &= (\widetilde{v}_{j_1}\widetilde{v}_{j_2} - \sigma^{j_1,j_2})\phi_\Sigma(\mathbf{v}),\\
\phi_\Sigma^{j_1,j_2,j_3}(\mathbf{v}) &= (\widetilde{v}_{j_1}\widetilde{v}_{j_2}\widetilde{v}_{j_3} - \langle 3\rangle\sigma^{j_1,j_2}\widetilde{v}_{j_3})\phi_\Sigma(\mathbf{v}),
\end{aligned}$$

where $\widetilde{v}_j = \sigma^{j,j'}v_{j'}$ (we used the standard summation convention instead of $\sum_{j'=1}^{p}\sigma^{j,j'}v_{j'}$), with $\sigma^{j,j'}$ being the (j,j')th element of $\mathbf{\Sigma}^{-1}$. Here, the notation $\langle n\rangle$ before a term with indices is a sum of n similar terms obtained by index permutation, e.g., $\langle 3\rangle\sigma^{j_1,j_2}\widetilde{v}_{j_3} = \sigma^{j_1,j_2}\widetilde{v}_{j_3} + \sigma^{j_1,j_3}\widetilde{v}_{j_2} + \sigma^{j_2,j_3}\widetilde{v}_{j_1}$. Note that

$$\phi_\Sigma^{j_1,\ldots,j_n}(\mathbf{v}) = \sum_{i=0}^{\lfloor n/2\rfloor} \Big\langle\binom{n}{2i}\Big\rangle_{j_1\cdots j_{2i}|j_{2i+1}\cdots j_n} \left[(-1)^i \Big\langle\frac{(2i)!}{2^i i!}\Big\rangle_{\text{all } i \text{ pairs}} \sigma^{j_1,j_2}\cdots\sigma^{j_{2i-1},j_{2i}}\right]\widetilde{v}_{j_{2i+1}}\cdots\widetilde{v}_{j_n}\phi_\Sigma(\mathbf{v})$$

(e.g., Barndorff-Nielsen and Cox, 1989, page 151, with somewhat different notation).

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Given $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_p)'$, it is well known (e.g., Anderson, 2003, page 82) that if $\mathbf{Z} \sim \mathrm{N}_p(\boldsymbol{\tau}, \boldsymbol{\Sigma})$, then, $\mathbf{Z}'\boldsymbol{\Sigma}^{-1}\mathbf{Z}$ has a noncentral χ^2-distribution with p degrees of freedom and noncentral parameter $\boldsymbol{\tau}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\tau}$;

$$\Pr[\mathbf{Z}'\boldsymbol{\Sigma}^{-1}\mathbf{Z} \le x] = \int_0^x g_p(t; \boldsymbol{\tau}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\tau})\, dt \quad \text{for } x > 0, \tag{1}$$

where

$$g_\nu(t; \omega^2) = \sum_{\ell=0}^{\infty} \frac{(\omega^2/2)^\ell \exp(-\omega^2/2)}{\ell!} \frac{t^{(\nu+2\ell)/2-1}\exp(-t/2)}{2^{(\nu+2\ell)/2}\Gamma((\nu+2\ell)/2)}$$

is the density of a noncentral χ^2-distribution with ν degrees of freedom and noncentral parameter ω^2. As usual, we write $G_\nu(x; \omega^2) = \int_0^x g_\nu(t; \omega^2)\, dt$ and $G_\nu^-(x; \omega^2) = 1 - G_\nu(x; \omega^2)$.

We are concerned with the integral of $\phi_\Sigma^{j_1,\ldots,j_q}(\mathbf{v})$ over the convex set $\{\mathbf{v} \in \mathbf{R}^p : (\mathbf{v}+\boldsymbol{\tau})'\boldsymbol{\Sigma}^{-1}(\mathbf{v}+\boldsymbol{\tau}) \le x\}$;

$$\int_{(\mathbf{v}+\tau)'\Sigma^{-1}(\mathbf{v}+\tau)\le x} \phi_\Sigma^{j_1,\ldots,j_q}(\mathbf{v})\, d\mathbf{v} = -\int_{\mathcal{D}_\tau(x)} \phi_\Sigma^{j_1,\ldots,j_q}(\mathbf{v})\, d\mathbf{v} \quad \text{for } q \in \mathbf{N}, \tag{2}$$

where $\mathcal{D}_\tau(x) = \{\mathbf{v} \in \mathbf{R}^p : (\mathbf{v}+\boldsymbol{\tau})'\boldsymbol{\Sigma}^{-1}(\mathbf{v}+\boldsymbol{\tau}) > x\}$ (we used $\int_{\mathbf{R}^p} \phi_\Sigma^{j_1,\ldots,j_q}(\mathbf{v})\, d\mathbf{v} = 0$; the orthogonality of the multivariate Hermite polynomial). Such integrals (2) for $q = 1, 2, 3, 4, 5, 6$ are implicitly used in the derivation of asymptotic expansions for the distributions of asymptotically chi-squared test statistics (e.g., Mukerjee, 1990a,b, Fujikoshi, 1997, Bravo, 2003, 2004, and Kakizawa, 2010a,b, 2012a,b, 2013). More precisely, we suppose that a random vector $\mathbf{U}^{(N)} = (U_1^{(N)}, \ldots, U_p^{(N)})' \xrightarrow{d} \mathrm{N}_p(\boldsymbol{\tau}, \boldsymbol{\Sigma})$ possesses cumulants

$$\begin{aligned}
\mathrm{cum}(U_j^{(N)}) &= \tau_j + \sum_{\ell\ge 1} N^{-\ell/2} c_j^{\langle\ell\rangle}, \\
\mathrm{cum}(U_{j_1}^{(N)}, U_{j_2}^{(N)}) &= \sigma_{j_1,j_2} + \sum_{\ell\ge 1} N^{-\ell/2} c_{j_1j_2}^{\langle\ell\rangle}, \\
\mathrm{cum}(U_{j_1}^{(N)}, \ldots, U_{j_s}^{(N)}) &= \sum_{\ell\ge s-2} N^{-\ell/2} c_{j_1\cdots j_s}^{\langle\ell\rangle}, \quad s \in \{3, 4, \ldots\},
\end{aligned}$$

where $c_\#^{\langle\ell\rangle}$'s are the coefficients of $N^{-\ell/2}$ in the cumulants. We write $e_\ell(\mathbf{z}) = \sum_{r=1}^{\ell+2} \frac{1}{r!} c_{j_1\cdots j_r}^{\langle\ell\rangle} z_{j_1} \ldots z_{j_r}$ for $\mathbf{z} = (z_1, \ldots, z_p)' \in \mathbf{C}^p$, and prepare the ordinary partial Bell polynomial $B_{\ell,k}^\circ = B_{\ell,k}^\circ(x_1, x_2, \ldots)$ for positive integers ℓ and k ($\ell \ge k$), defined as the formal series expansion $(\sum_{\ell\ge1} x_\ell y^\ell)^k = \sum_{\ell\ge k} B_{\ell,k}^\circ y^\ell$, i.e., $B_{\ell,k}^\circ = \sum \frac{\ell!}{i_1!i_2!\ldots} x_1^{i_1} x_2^{i_2} \cdots$ (see Comtet, 1974, page 136), where the summation takes over all nonnegative integers $i_1, i_2, \ldots$, such that $i_1 + i_2 + i_3 + \cdots = k$ and $i_1 + 2i_2 + 3i_3 + \cdots = \ell$ (e.g., $B_{1,1}^\circ = x_1$, $B_{2,1}^\circ = x_2$, $B_{2,2}^\circ = x_1^2$). Then, as in Taniguchi (1991, page 14) (see also Withers and Nadarajah, 2010), the characteristic function of $\mathbf{U}^{(N)}$ is formally expanded as

$$\begin{aligned}
&\exp\Big\{\tau_j(\mathrm{i}t_j) + \frac{1}{2!}\sigma_{j_1j_2}(\mathrm{i}t_{j_1})(\mathrm{i}t_{j_2}) + \sum_{s\ge1}\frac{1}{s!}\sum_{\ell\ge\max(1,s-2)} N^{-\ell/2} c_{j_1\cdots j_s}^{\langle\ell\rangle}(\mathrm{i}t_{j_1})\ldots(\mathrm{i}t_{j_s})\Big\} \\
&= \exp\Big\{\mathrm{i}t_j\tau_j - \frac{1}{2}\sigma_{j_1j_2}t_{j_1}t_{j_2} + \sum_{\ell\ge1} N^{-\ell/2} e_\ell(\mathrm{i}\mathbf{t})\Big\} \\
&= \exp\Big(\mathrm{i}t_j\tau_j - \frac{1}{2}\sigma_{j_1j_2}t_{j_1}t_{j_2}\Big)\Big[1 + \sum_{k\ge1}\frac{1}{k!}\sum_{\ell\ge k} N^{-\ell/2} B_{\ell,k}^\circ(e_1(\mathrm{i}\mathbf{t}), e_2(\mathrm{i}\mathbf{t}), \ldots)\Big] \\
&= \exp\Big(\mathrm{i}t_j\tau_j - \frac{1}{2}\sigma_{j_1j_2}t_{j_1}t_{j_2}\Big)\Big[1 + \sum_{\ell\ge1} N^{-\ell/2}\sum_{k=1}^{\ell}\frac{1}{k!} B_{\ell,k}^\circ(e_1(\mathrm{i}\mathbf{t}), e_2(\mathrm{i}\mathbf{t}), \ldots)\Big] \\
&= \exp\Big(\mathrm{i}t_j\tau_j - \frac{1}{2}\sigma_{j_1j_2}t_{j_1}t_{j_2}\Big)\Big[1 + \sum_{\ell\ge1} N^{-\ell/2} C_\ell^\circ(\mathrm{i}\mathbf{t})\Big] \quad (\mathrm{i} = \sqrt{-1} \text{ and } \mathbf{t} = (t_1, \ldots, t_p)' \in \mathbf{R}^p),
\end{aligned}$$

where $C_\ell^\circ(\mathbf{z}) = \sum_{k=1}^{\ell} \frac{1}{k!} B_{\ell,k}^\circ(e_1(\mathbf{z}), e_2(\mathbf{z}), \ldots)$ is a polynomial of degree 3ℓ (with $C_\ell^\circ(\mathbf{0}_p) = 0$); note that $C_1^\circ(\mathbf{z}) = e_1(\mathbf{z})$ and $C_2^\circ(\mathbf{z}) = e_2(\mathbf{z}) + e_1^2(\mathbf{z})/2$. It follows that the distribution of $\mathbf{U}^{(N)}$ over a certain Borel set $A \subset \mathbf{R}^p$ may be approximated as

$$\Pr[\mathbf{U}^{(N)} \in A] = \int_A \Big[1 + \sum_{\ell=1}^{r-2} N^{-\ell/2} C_\ell^\circ\Big(-\frac{\partial}{\partial v_{j_1}}, \ldots, -\frac{\partial}{\partial v_{j_p}}\Big)\Big]\phi_\Sigma(\mathbf{v}-\boldsymbol{\tau})\, d\mathbf{v} + o(N^{-(r-2)/2}).$$

Especially, the choice $A = \{\mathbf{v} \in \mathbf{R}^p : \mathbf{v}'\boldsymbol{\Sigma}^{-1}\mathbf{v} \le x\}$ yields

$$\Pr[(\mathbf{U}^{(N)})'\boldsymbol{\Sigma}^{-1}\mathbf{U}^{(N)} \le x] = G_p(x; \boldsymbol{\tau}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\tau}) - \sum_{\ell=1}^{r-2} N^{-\ell/2}\int_{\mathcal{D}_\tau(x)} C_\ell^\circ\Big(-\frac{\partial}{\partial v_{j_1}}, \ldots, -\frac{\partial}{\partial v_{j_p}}\Big)\phi_\Sigma(\mathbf{v})\, d\mathbf{v} + o(N^{-(r-2)/2}). \tag{3}$$