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On infinite dimensional periodically correlated random fields: Spectrum and evolutionary spectra

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ABSTRACT

Infinite dimensional periodically correlated (PC) random fields are studied in spectral domain. A spectral characterization is given and harmonizability is established. The covariance operator is characterized where it is observed that an infinite dimensional PC field is a two-dimensional Fourier transform of a spectral random measure. Also, an evolutionary spectral representation and a space-dependent spectral density are given.

1. Introduction

Periodically Correlated (PC) random processes, also known as cyclostationary processes, are non-stationary processes in which the non-stationarity occurs in periodic mean and covariance that makes possible a manageable spectral theory. These processes which were introduced by Gladyshev (1961), have enjoyed tremendous attention from different authors. In order to elaborate the theory of univariate PC processes see Hurd (1974), Soltani and Shishebor (1998, 1999), Hurd and Miamee (2007) and Makagon and Miamee (2013). The importance of this class of nonstationary processes is due to a variety of applications in different areas of sciences and engineering. The bibliography by Gardner et al. (2006) provides a comprehensive list of references on these processes and their various applications. In the context of random fields, this was initiated in Hurd et al. (2004) who considered univariate PC random fields and extended the basic results of Gladyshev (1961) to fields. These processes occur for example in electromagnetic fields, the microwave electronics, acoustic fields produced by radiated PC acoustic sources and bidimensional ocean waves. A characterization of covariance operator for second order locally square integrable PC random fields was provided by Gaspar (2004). Hilbertian spatial PC autoregressive models, were introduced by Haghbin et al. (2014). They studied necessary conditions to have a moving average representation and applied the proposed model to predict pixels of invisible infrared sequential satellite images. The covariance function, the notion of the spectrum, and the structure of random fields indexed on a locally compact Abelian group was studied by Dehay et al. (2014).

The concept of *evolutionary spectra for non-stationary processes was* introduced by Priestley (1965). The evolutionary spectrum is a spectral function which is time dependent and has a physical interpretation as local energy distribution over frequency. This concept generalizes the usual definition of spectra for stationary processes. Soltani and Shishebor (2007) portrayed an evolutionary spectral representation for infinite dimensional PC process. Recent technological progress in data collection and storage allow to record data at high frequency. To benefit from this huge information, we need to appropriate

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infinite dimensional theory. This fact and the importance of evolutionary spectrum constitute the main motivation of this paper, which addresses the problem of infinite dimensional PC random fields and aims to derive a spectral characterization and an evolutionary spectral measure for such random fields. Although, the idea of evolutionary spectral measure has been extended to infinite dimensional PC processes, to the best of our knowledge, this idea has not been investigated in the PC random fields, even in the usual univariate random fields setting. The significance of this representation lies on the fact that the law of the process is governed by a deterministic operator and a positive measure. For corresponding results in the context of time series, see Soltani and Shishebor (2007) and in the context of univariate random fields, see Hurd et al. (2004).

This article is organized as follows. In Section 2, we introduce notations and terminology used throughout the paper. In Section 3, a spectral characterization of infinite dimensional PC random fields is developed and harmonizability is established. In Section 4, we characterize the covariance operator of infinite dimensional PC random fields. This is proved that an infinite dimensional PC random field is the two-dimensional Fourier transform of a spectral random measure, which indeed is an operator valued measure. An evolutionary spectral representation for an infinite dimensional PC random field is given in Section 5.

2. Preliminary definitions

Let H be a Hilbert Space with inner product $\langle\cdot,\cdot\rangle_H$ and norm $\|\cdot\|_H$. Let $\mathcal{L}(H)$ denote the Banach space of bounded linear operators from H to H. For $x, y \in H$, the notation $x\otimes y$ means tensor product which is defined by $x\otimes y\,(h) = \langle x, h\rangle_H\, y,\ h \in H$. The probability space is denoted by $(\Omega, \mathcal{F}, P)$ and let L^2_H stand for the Hilbert space of zero mean H-valued random variables on $(\Omega, \mathcal{F}, P)$ with $E\left(\|X\|^2_H\right) < \infty$ and inner product $\langle X, Y\rangle_{L^2_H} \equiv E\,\langle X, Y\rangle_H$ for $X, Y \in L^2_H$. For X in L^2_H, $\mathbb{E}(X) \equiv \int_\Omega X(\omega)dP(\omega)$, where the integration is in the sense of Bochner integral, see Dunford and Schwartz (1958). The cross covariance operator is defined as $C_{X,Y} = \mathbb{E}\,(X \otimes Y)$ and the covariance operator is denoted by $C_X = C_{X,X}$. Also let L_2 be the Hilbert space of zero mean real-valued random variables on $(\Omega, \mathcal{F}, P)$ with $E\left(X^2\right) < \infty$ and inner product $\langle X, Y\rangle_{L_2} \equiv EXY$ for $X, Y \in L_2$.

For notational convenience, we define $\mathbf{t} = (t_1, t_2)$ for any $\mathbf{t} \in \mathbb{R}^2$. For $\mathbf{s}, \mathbf{t} \in \mathbb{R}^2$ and $\mathbf{n} \in \mathbb{N}^2$, let $\mathbf{s}\odot\mathbf{t} = (s_1t_1, s_2t_2)$, $\langle \mathbf{s}, \mathbf{t}\rangle = s_1t_1 + s_2t_2$ and $\mathbf{t}/\mathbf{n} = \left(\frac{t_1}{n_1}, \frac{t_2}{n_2}\right)$. For $\mathbf{T} \in \mathbb{N}^2$, the sequence $\left\{a_\mathbf{t}, \mathbf{t} \in \mathbb{Z}^2\right\}$ of operators in $\mathcal{L}(H)$ is called $\mathbf{T}$-periodic if $a_{\mathbf{t}+\mathbf{n}\odot\mathbf{T}} = a_\mathbf{t}$ for all $\mathbf{n} \in \mathbb{Z}^2$, where the equality means in the sense of $\mathcal{L}(H)$ norm.

Definition 1. A zero mean second-order H-valued random field $\mathbf{X} = \left\{X_\mathbf{t}, \mathbf{t} \in \mathbb{Z}^2\right\}$ is called Periodically Correlated with period $\mathbf{T}$ ($\mathbf{T}$-HPC) if there exists $\mathbf{T} \in \mathbb{N}^\mathbf{2}$ such that for each fixed $\mathbf{s}, \mathbf{t} \in \mathbb{Z}^2$,

$$\mathbb{E}\,(X_\mathbf{s} \otimes X_\mathbf{t}) = \mathbb{E}\,(X_{\mathbf{s}+\mathbf{n}\odot\mathbf{T}} \otimes X_{\mathbf{t}+\mathbf{n}\odot\mathbf{T}}), \quad \mathbf{n} \in \mathbb{Z}^2. \tag{2.1}$$

We refer to $\mathbf{T} = (T_1, T_2)$ as the period if T_1 and T_2 are the smallest positive integers satisfying (2.1). Note that if $\mathbf{T} = (1, 1)$, we get stationary random field.

Definition 2. A zero mean second-order H-valued random field $\mathbf{P} = \left\{P_\mathbf{t}, \mathbf{t} \in \mathbb{Z}^2\right\}$ is called periodic field if there exists a $\mathbf{T} \in \mathbb{N}^\mathbf{2}$ such that for each $\mathbf{t}, \mathbf{n} \in \mathbb{Z}^2$, $P_\mathbf{t} = P_{\mathbf{t}+\mathbf{n}\odot\mathbf{T}}$, where the equality means with probability one.

From now on, we suppose $T_1, T_2 \in \mathbb{N}$, $\mathbf{T} = (T_1, T_2)$, $\mathbf{T}_1 = (T_1, 1)$ and $\mathbf{T}_2 = (1, T_2)$. Define $D_\mathbf{T} = \left\{(i, j),\ i = 0, \ldots, T_1-1,\ j = 0, \ldots, T_2 - 1\right\}$ and the function $\omega : D_\mathbf{T} \to \{1, \ldots, T_1T_2\}$ by $\omega(\mathbf{t}) = t_2T_1 + t_1 + 1$. For any $x \in H$ and $\mathbf{t} \in D_\mathbf{T}$, let $\tilde{x}\,(\mathbf{t})$ be a vector in $H^{T_1T_2}$ with all coordinates are zero except $\omega\,(\mathbf{t})$-th, which is x. The *time* domain of $\mathbf{X}$ is denoted by $\mathcal{H}_X = \overline{\mathrm{sp}}\left\{X^x_\mathbf{n},\ x \in H,\ \mathbf{n} \in \mathbb{Z}^2\right\}$, where $X^x_\mathbf{n} = \langle X_\mathbf{n}, x\rangle_H$ and the closure is with respect to the L_2 norm. It is well known that for an H-valued stationary random field $Y = \left\{Y_\mathbf{n}, \mathbf{n} \in \mathbb{N}^2\right\}$, the spectral representation

$$Y_\mathbf{n} = \int_{[0,2\pi]^2} e^{-i\langle\boldsymbol{\lambda},\mathbf{n}\rangle}\,\Phi\,(d\boldsymbol{\lambda}), \quad \mathbf{n} \in \mathbb{N}^2, \tag{2.2}$$

in the sense that

$$Y^x_\mathbf{n} = \int_{[0,2\pi]^2} e^{-i\langle\mathbf{n},\boldsymbol{\lambda}\rangle}\,\Phi\,(d\boldsymbol{\lambda})\,x, \tag{2.3}$$

and

$$EY^x_\mathbf{m}\overline{Y^y_\mathbf{n}} = \int_{[0,2\pi]^2} e^{-i\langle\mathbf{m}-\mathbf{n},\boldsymbol{\lambda}\rangle} E\left(\Phi\,(d\boldsymbol{\lambda})\,x\overline{\Phi\,(d\boldsymbol{\lambda})\,y}\right), \tag{2.4}$$

is fulfilled. In (2.2), $\Phi\,(d\boldsymbol{\lambda})$ is a spectral random measure with orthogonal increments on H, i.e., Φ is a finitely additive set function on the Borel subsets of $[0, 2\pi]^2 = [0, 2\pi] \times [0, 2\pi]$ with values in the space of bounded linear transformations from H into L^2_H. The spectral distribution of a stationary field Y is an $\mathcal{L}(H)$-valued measure F for which

$$\langle x, F(d\boldsymbol{\lambda})y\rangle_H = E\left(\Phi\,(d\boldsymbol{\lambda})\,x\overline{\Phi\,(d\boldsymbol{\lambda})\,y}\right).$$

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